

Chapter 4

Lognormal random variables. Risk-neutral pricing.

4.1 Solutions to Chapter 4 Exercises

Problem 1: Let $X_1 = Z$ and $X_2 = -Z$ be two independent random variables, where Z is the standard normal variable. Show that $X_1 + X_2$ is a normal variable of mean 0 and variance 2, i.e., $X_1 + X_2 = \sqrt{2}Z$.

Solution: Recall that if X_1 and X_2 are independent normal random variables with mean and variance μ_1 and σ_1^2 , and μ_2 and σ_2^2 , respectively, then $X_1 + X_2$ is a normal variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$, and

$$X_1 + X_2 = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2} Z.$$

For $X_1 = Z$ and $X_2 = -Z$, it follows that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. We conclude that

$$E[X] = \mu_1 + \mu_2 = 0 \quad \text{and} \quad \text{var}(X) = \sigma_1^2 + \sigma_2^2 = 2,$$

and therefore

$$X = X_1 + X_2 = \sqrt{2}Z. \quad \square$$

Problem 2: Assume that the normal random variables X_1, X_2, \dots, X_n of mean μ and variance σ^2 are uncorrelated, i.e., $\text{cov}(X_i, X_j) = 0$, for all $1 \leq i \neq j \leq n$. (This happens, e.g., if X_1, X_2, \dots, X_n are independent.) If $S_n = \frac{1}{n} \sum_{i=1}^n X_i$ is the average of the variables $X_i, i = 1 : n$, show that

$$E[S_n] = \mu \quad \text{and} \quad \text{var}(S_n) = \frac{\sigma^2}{n}.$$

Solution: Recall that, for $c_i \in \mathbb{R}$,

$$E \left[\sum_{i=1}^n c_i X_i \right] = \sum_{i=1}^n c_i E[X_i];$$

$$\text{var} \left(\sum_{i=1}^n c_i X_i \right) = \sum_{i=1}^n c_i^2 \text{var}(X_i) + 2 \sum_{1 \leq i < j \leq n} c_i c_j \text{cov}(X_i, X_j).$$

Therefore,

$$\begin{aligned} E[S_n] &= E \left[\frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n E[X_i] = \frac{1}{n} \cdot n\mu = \mu; \\ \text{var}(S_n) &= \text{var} \left(\frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2} \sum_{i=1}^n \text{var}(X_i) + \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \text{cov}(X_i, X_j) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{var}(X_i) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}, \end{aligned}$$

since $\text{cov}(X_i, X_j) = 0$ for all $1 \leq i \neq j \leq n$. \square

Problem 3: Assume we have a one period binomial model for the evolution of the price of an underlying asset between time t and time $t + \delta t$:

If $S(t)$ is the price of the asset at time t , then the price $S(t + \delta t)$ of the asset at time $t + \delta t$ will be either $S(t)u$, with (risk-neutral) probability p , or $S(t)d$, with probability $1 - p$. Assume that $u > 1$ and $d < 1$.

Show that

$$E_{RN}[S(t + \delta t)] = (pu + (1 - p)d) S(t); \quad (4.1)$$

$$E_{RN}[S^2(t + \delta t)] = (pu^2 + (1 - p)d^2) S^2(t). \quad (4.2)$$

Solution: We can regard $S(t + \delta t)$ as a random variable over the probability space $\{U, D\}$ of the possible moves of the price of the asset from time t to time $t + \delta t$ endowed with the risk-neutral probability function $P : \{U, D\} \rightarrow [0, 1]$ with $P(U) = p$ and $P(D) = 1 - p$. Then $S(t + \delta t)$ is given by

$$S(t + \delta t)(U) = S(t)u; \quad S(t + \delta t)(D) = S(t)d.$$

Then, by definition,

$$\begin{aligned} E_{RN}[S(t + \delta t)] &= P(U) \cdot S(t + \delta t)(U) + P(D) \cdot S(t + \delta t)(D) \\ &= (pu + (1 - p)d) S(t); \\ E_{RN}[S^2(t + \delta t)] &= P(U) \cdot (S(t + \delta t)(U))^2 + P(D) \cdot (S(t + \delta t)(D))^2 \\ &= (pu^2 + (1 - p)d^2) S^2(t). \quad \square \end{aligned}$$

Problem 4: If the price $S(t)$ of a non-dividend paying asset has lognormal distribution with drift r and volatility σ , show that

$$E_{RN}[S(t + \delta t)] = e^{r\delta t} S(t); \quad (4.3)$$

$$E_{RN}[S^2(t + \delta t)] = e^{(2r + \sigma^2)\delta t} S^2(t). \quad (4.4)$$

Solution 1: If the price $S(t)$ of the non-dividend paying asset has lognormal distribution with drift r and volatility σ , then $\frac{S(t + \delta t)}{S(t)}$ is a lognormal variable given by

$$\ln \left(\frac{S(t + \delta t)}{S(t)} \right) = \left(r - \frac{\sigma^2}{2} \right) \delta t + \sigma \sqrt{\delta t} Z.$$

Recall that, if $\ln(Y) = \mu + \tilde{\sigma}Z$ is a lognormal random variable with parameters μ and $\tilde{\sigma}$, the expected value and variance of Y are

$$E[Y] = \exp \left(\mu + \frac{\tilde{\sigma}^2}{2} \right);$$

$$\text{var}(Y) = \exp(2\mu + \tilde{\sigma}^2) \left(e^{\tilde{\sigma}^2} - 1 \right).$$

If $Y = \frac{S(t + \delta t)}{S(t)}$, then $\mu = \left(r - \frac{\sigma^2}{2} \right) \delta t$ and $\tilde{\sigma} = \sigma \sqrt{\delta t}$ and therefore

$$E \left[\frac{S(t + \delta t)}{S(t)} \right] = \exp \left(\left(r - \frac{\sigma^2}{2} \right) \delta t + \frac{(\sigma \sqrt{\delta t})^2}{2} \right) = e^{r\delta t}; \quad (4.5)$$

$$\begin{aligned} \text{var} \left(\frac{S(t + \delta t)}{S(t)} \right) &= \exp \left(2 \left(r - \frac{\sigma^2}{2} \right) \delta t + (\sigma \sqrt{\delta t})^2 \right) \cdot \left(e^{(\sigma \sqrt{\delta t})^2} - 1 \right) \\ &= e^{2r\delta t} \left(e^{\sigma^2 \delta t} - 1 \right). \end{aligned} \quad (4.6)$$

Note that,

$$E \left[\frac{S(t + \delta t)}{S(t)} \right] = \frac{1}{S(t)} \cdot E[S(t + \delta t)] \quad (4.8)$$

$$\text{var} \left(\frac{S(t + \delta t)}{S(t)} \right) = \frac{1}{S^2(t)} \cdot \text{var}(S(t + \delta t)). \quad (4.9)$$

From (4.5) and (4.8), and from (4.7) and (4.9), respectively, we conclude that

$$\begin{aligned} E[S(t + \delta t)] &= e^{r\delta t} S(t); \\ \text{var}(S(t + \delta t)) &= e^{2r\delta t} \left(e^{\sigma^2 \delta t} - 1 \right) S^2(t). \end{aligned} \quad (4.10)$$

Note that

$$\begin{aligned}\text{var}(S(t + \delta t)) &= E[S^2(t + \delta t)] - (E[S(t + \delta t)])^2 \\ &= E[S^2(t + \delta t)] - e^{2r\delta t} S^2(t).\end{aligned}\quad (4.11)$$

From (4.10) and (4.11) it follows that

$$E[S^2(t + \delta t)] = \text{var}(S(t + \delta t)) + e^{2r\delta t} S^2(t) = e^{2r\delta t + \sigma^2 \delta t} S^2(t),$$

which is what we wanted to show.

Solution 2: Note that $S(t + \delta t)$ can be written as a function of the standard normal variable Z as follows:

$$S(t + \delta t) = S(t) \exp\left(\left(r - \frac{\sigma^2}{2}\right) \delta t + \sigma \sqrt{\delta t} Z\right).$$

Then,

$$\begin{aligned}E[S(t + \delta t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S(t) \exp\left(\left(r - \frac{\sigma^2}{2}\right) \delta t + \sigma \sqrt{\delta t} x\right) e^{-\frac{x^2}{2}} dx \\ &= S(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(r\delta t - \frac{\sigma^2 \delta t}{2} + \sigma \sqrt{\delta t} x - \frac{x^2}{2}\right) dx \\ &= S(t) e^{r\delta t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - \sigma \sqrt{\delta t})^2}{2}\right) dx \\ &= S(t) e^{r\delta t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= S(t) e^{r\delta t},\end{aligned}$$

where we used the substitution $y = x - \sigma \sqrt{\delta t}$ and the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = 1,$$

since $\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ is the density function of the standard normal variable.

Similarly, we obtain that

$$\begin{aligned}E[S^2(t + \delta t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S^2(t) \exp\left(2\left(r - \frac{\sigma^2}{2}\right) \delta t + 2\sigma \sqrt{\delta t} x\right) e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S^2(t) \exp\left(2r\delta t - \sigma^2 \delta t + 2\sigma \sqrt{\delta t} x - \frac{x^2}{2}\right) dx\end{aligned}$$

$$\begin{aligned}
&= S^2(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(2r\delta t + \sigma^2\delta t - \frac{(x - 2\sigma\sqrt{\delta t})^2}{2}\right) dx \\
&= S^2(t)e^{(2r+\sigma^2)\delta t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - 2\sigma\sqrt{\delta t})^2}{2}\right) dx \\
&= S^2(t)e^{(2r+\sigma^2)\delta t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds \\
&= S^2(t)e^{(2r+\sigma^2)\delta t},
\end{aligned}$$

the substitution $s = x - 2\sigma\sqrt{\delta t}$ was used above. \square

Problem 5: The results of the previous two exercises can be used to calibrate a binomial tree model to a lognormally distributed process. This means finding the up and down factors u and d , and the risk-neutral probability p (of going up) such that the values of $E_{RN}[S(t+\delta t)]$ and $E_{RN}[S^2(t+\delta t)]$ given by (4.1) and (4.2) coincide with the values (4.3) and (4.4) for the lognormal model.

In other words, we are looking for u , d , and p such that

$$pu + (1-p)d = e^{r\delta t}; \quad (4.12)$$

$$pu^2 + (1-p)d^2 = e^{(2r+\sigma^2)\delta t}. \quad (4.13)$$

Since there are two constraints and three unknowns, the solution will not be unique.

(i) Show that (4.12–4.13) are equivalent to

$$p = \frac{e^{r\delta t} - d}{u - d}; \quad (4.14)$$

$$(e^{r\delta t} - d)(u - e^{r\delta t}) = e^{2r\delta t}(e^{\sigma^2\delta t} - 1). \quad (4.15)$$

(ii) Derive the Cox–Ross–Rubinstein parametrization for a binomial tree, by solving (4.14–4.15) with the additional condition that

$$ud = 1.$$

Show that the solution can be written as

$$p = \frac{e^{r\delta t} - d}{u - d}; \quad u = A + \sqrt{A^2 - 1}; \quad d = A - \sqrt{A^2 - 1},$$

where

$$A = \frac{1}{2} \left(e^{-r\delta t} + e^{(r+\sigma^2)\delta t} \right). \quad (4.16)$$

Solution: (i) Formula (4.14) can be obtained by solving the linear equation (4.12) for p .

To obtain (4.15), we first square formula (4.12) to obtain

$$p^2u^2 + 2p(1-p)ud + (1-p)^2d^2 = e^{2r\delta t}$$

and subtract this from (4.13). We find that

$$p(1-p)u^2 - 2p(1-p)ud + p(1-p)d^2 = e^{(2r+\sigma^2)\delta t} - e^{2r\delta t},$$

which can be written as

$$p(1-p)(u-d)^2 = e^{2r\delta t} (e^{\sigma^2\delta t} - 1). \quad (4.17)$$

Using formula (4.14) for p , it is easy to see that

$$p(1-p) = \frac{(e^{r\delta t} - d)(u - e^{r\delta t})}{(u-d)^2}. \quad (4.18)$$

From (4.17) and (4.18), we conclude that

$$(e^{r\delta t} - d)(u - e^{r\delta t}) = e^{2r\delta t} (e^{\sigma^2\delta t} - 1).$$

(ii) By multiplying out (4.15) and using the fact that $ud = 1$, we obtain that

$$ue^{r\delta t} - 1 - e^{2r\delta t} + de^{r\delta t} = e^{(2r+\sigma^2)\delta t} - e^{2r\delta t}. \quad (4.19)$$

After canceling out the term $-e^{2r\delta t}$, we divide (4.19) by $e^{r\delta t}$ and obtain

$$u + d - e^{-r\delta t} = e^{(r+\sigma^2)\delta t},$$

which can be written as

$$u + \frac{1}{u} - \left(e^{-r\delta t} + e^{(r+\sigma^2)\delta t} \right) = u + \frac{1}{u} - 2A = 0;$$

cf. (4.16) for the definition of A .

In other words, u is a solution of the quadratic equation

$$u^2 - 2Au + 1 = 0, \quad (4.20)$$

which has two solutions, $A + \sqrt{A^2 - 1}$ and $A - \sqrt{A^2 - 1}$. Since $u > 1$, we conclude that

$$u = A + \sqrt{A^2 - 1};$$

the other solution of the quadratic equation (4.20) corresponds to the value of d , since

$$d = \frac{1}{u} = \frac{1}{A + \sqrt{A^2 - 1}} = A - \sqrt{A^2 - 1}. \quad \square$$

Problem 6: Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, while the series $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$ are divergent, i.e., equal to ∞ .

Note: It is known that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

and

$$\lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln(n) \right) = \gamma,$$

where $\gamma \approx 0.57721$ is called Euler's constant.

Solution: Since all the terms of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ are positive, it is enough to show that the partial sums

$$\sum_{k=1}^n \frac{1}{k^2}$$

are uniformly bounded, in order to conclude that the series is convergent. This can be seen as follows:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k^2} &= 1 + \sum_{k=2}^n \frac{1}{k^2} \leq 1 + \sum_{k=2}^n \frac{1}{k(k-1)} = 1 + \sum_{k=2}^n \frac{1}{k-1} - \frac{1}{k} \\ &= 1 + \left(1 - \frac{1}{n}\right) < 2, \quad \forall n \geq 2. \end{aligned}$$

To show that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, we will prove that

$$\ln(n) + \frac{1}{n} < \sum_{k=1}^n \frac{1}{k} < \ln(n) + 1, \quad \forall n \geq 1. \quad (4.21)$$

The integral of the function $f(x) = \frac{1}{x}$ over the interval $[1, n]$ can be approximated from above and below as follows: Note that

$$\int_1^n \frac{1}{x} dx = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x} dx.$$

Since $f(x) = \frac{1}{x}$ is a decreasing function, it is easy to see that

$$\frac{1}{k+1} < f(x) < \frac{1}{k}, \quad \forall x \in (k, k+1).$$

Then,

$$\begin{aligned} \int_1^n \frac{1}{x} dx &= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x} dx > \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{k+1} dx \\ &= \sum_{k=1}^{n-1} \frac{1}{k+1} = -1 + \sum_{k=1}^n \frac{1}{k} \end{aligned} \quad (4.22)$$

$$\begin{aligned} \int_1^n \frac{1}{x} dx &= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x} dx < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{k} dx \\ &= \sum_{k=1}^{n-1} \frac{1}{k} = -\frac{1}{n} + \sum_{k=1}^n \frac{1}{k} \end{aligned} \quad (4.23)$$

The inequality (4.21) follows from (4.22) and (4.23), since

$$\int_1^n \frac{1}{x} dx = \ln(n).$$

In a similar fashion, by considering the integral of $\frac{1}{x \ln(x)}$ over the interval $[2, n]$, we can show that

$$\ln(\ln(n)) - \ln(\ln(2)) + \frac{1}{n \ln(n)} < \sum_{k=2}^n \frac{1}{k \ln(k)}, \quad \forall n \geq 2; \quad (4.24)$$

$$\sum_{k=2}^n \frac{1}{k \ln(k)} < \ln(\ln(n)) - \ln(\ln(2)) + \frac{1}{2 \ln(2)}, \quad \forall n \geq 2, \quad (4.25)$$

and conclude that the series $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$ is divergent.

For example, (4.24) can be proved as follows:

$$\int_2^n \frac{1}{x \ln(x)} dx = \sum_{k=2}^{n-1} \int_k^{k+1} \frac{1}{x \ln(x)} dx < \sum_{k=2}^{n-1} \int_k^{k+1} \frac{1}{k \ln(k)} dx = \sum_{k=2}^{n-1} \frac{1}{k \ln(k)},$$

which is equivalent to

$$\int_2^n \frac{1}{x \ln(x)} dx + \frac{1}{n \ln(n)} < \sum_{k=2}^n \frac{1}{k \ln(k)}. \quad (4.26)$$

Since

$$\int_2^n \frac{1}{x \ln(x)} dx = \ln(\ln(n)) - \ln(\ln(2)), \quad (4.27)$$

we conclude from (4.26) and (4.27) that

$$\ln(\ln(n)) - \ln(\ln(2)) + \frac{1}{n \ln(n)} < \sum_{k=2}^n \frac{1}{k \ln(k)},$$

which is the same as (4.24). \square

Problem 7: Find the radius of convergence R of the power series

$$\sum_{k=2}^{\infty} \frac{x^k}{k \ln(k)},$$

and investigate what happens at the points x where $|x| = R$.

Solution: It is easy to see that

$$\sum_{k=2}^{\infty} \frac{x^k}{k \ln(k)} = \sum_{k=2}^{\infty} a_k x^k, \quad (4.28)$$

with $a_k = \frac{1}{k \ln(k)}$, $k \geq 2$. Note that

$$\lim_{k \rightarrow \infty} |a_k|^{1/k} = \lim_{k \rightarrow \infty} \left(\frac{1}{k \ln(k)} \right)^{1/k} = 1. \quad (4.29)$$

Recall that, if $\lim_{k \rightarrow \infty} |a_k|^{1/k}$ exists, the radius of convergence of the series $\sum_{k=2}^{\infty} a_k x^k$ is given by

$$R = \frac{1}{\lim_{k \rightarrow \infty} |a_k|^{1/k}}. \quad (4.30)$$

From (4.29) and (4.30) the radius of convergence of the series (4.28) is $R = 1$. We conclude that the series is convergent if $|x| < 1$, and not convergent if $|x| > 1$.

If $x = -1$, the series becomes $\sum_{k=1}^{\infty} \frac{(-1)^k}{k \ln(k)}$. Since the terms $\frac{(-1)^k}{k \ln(k)}$ have alternating signs and decrease in absolute value to 0, the series is convergent.

If $x = 1$, the series becomes $\sum_{k=1}^{\infty} \frac{1}{k \ln(k)}$, which was shown to be divergent in Problem 6 of this chapter. \square

Problem 8: Consider a put option with strike 55 and maturity 4 months on a non-dividend paying asset with spot price 60 which follows a lognormal

model with drift $\mu = 0.1$ and volatility $\sigma = 0.3$. Assume that the risk-free rate is constant equal to 0.05.

- (i) Find the probability that the put will expire in the money.
- (ii) Find the risk-neutral probability that the put will expire in the money.
- (iii) Compute $N(-d_2)$.

Solution: (i) The probability that the put option will expire in the money is equal to the probability that the spot price at maturity is lower than the strike price, i.e., to $P(S(T) < K)$. Recall that

$$\ln\left(\frac{S(T)}{S(0)}\right) = \left(\mu - q - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z.$$

Then,

$$\begin{aligned} P(S(T) < K) &= P\left(\frac{S(T)}{S(0)} < \frac{K}{S(0)}\right) = P\left(\ln\left(\frac{S(T)}{S(0)}\right) < \ln\left(\frac{K}{S(0)}\right)\right) \\ &= P\left(\left(\mu - q - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z < \ln\left(\frac{K}{S(0)}\right)\right) \\ &= P\left(Z < \frac{\ln\left(\frac{K}{S(0)}\right) - \left(\mu - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ &= N\left(\frac{\ln\left(\frac{K}{S(0)}\right) - \left(\mu - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

For $S = 60$, $K = 55$, $T = 1/3$, $\mu = 0.1$, $q = 0$, $\sigma = 0.3$, and $r = 0.05$, we obtain that the probability that the put will expire in the money is 0.271525, i.e., 27.1525%.

(ii) The risk-neutral probability that the put option will expire in the money is obtained just like the probability that the put expires in the money, by substituting the risk-free rate r for μ , i.e.,

$$\begin{aligned} P_{RN}(S(T) < K) &= P\left(Z < \frac{\ln\left(\frac{K}{S(0)}\right) - \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ &= 0.304331 = 30.43\%. \end{aligned} \tag{4.31}$$

(iii) Recall that

$$d_2 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}. \tag{4.32}$$

Then, $d_2 = 0.511983$, and

$$N(-d_2) = 0.304331,$$

which is the same as the risk-neutral probability that the put option will expire in the money; cf. (4.31).

To understand this result, note that

$$P_{RN}(S(T) < K) = P\left(Z < -\frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = N(-d_2);$$

cf. (4.31) and (4.32). \square

Problem 9: (i) Consider an at-the-money call on a non-dividend paying asset; assume the Black-Scholes framework. Show that the Delta of the option is always greater than 0.5.

(ii) If the underlying asset pays dividends at the continuous rate q , when is the Delta of an at-the-money call less than 0.5?

Note: For most cases, the Delta of an at-the-money call option is close to 0.5.

Solution: (i) Recall that the Delta of a call option is given by

$$\Delta(C) = e^{-qT}N(d_1) = e^{-qT}N\left(\frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

For an at-the-money call on a non-dividend paying asset, i.e., for $K = S$ and $q = 0$, we find that

$$\Delta(C) = N(d_1) = N\left(\frac{\left(r + \frac{\sigma^2}{2}\right)\sqrt{T}}{\sigma}\right) \geq N(0) = 0.5.$$

(ii) If the underlying asset pays dividends at the continuous rate q , the Delta of an ATM call is

$$\Delta(C) = e^{-qT}N(d_1) = e^{-qT}N\left(\frac{\left(r - q + \frac{\sigma^2}{2}\right)\sqrt{T}}{\sigma}\right).$$

For a fixed risk-free rate r and fixed maturity T , we conclude that $\Delta(C) < 0.5$ if and only if the dividend yield q and the volatility σ of the underlying asset

satisfy the following condition:

$$N\left(\frac{\left(r - q + \frac{\sigma^2}{2}\right)\sqrt{T}}{\sigma}\right) < 0.5 e^{qT}.$$

This happens, for example, if $r = q$ and T is large enough, since

$$\lim_{T \rightarrow \infty} N\left(\frac{\sigma\sqrt{T}}{2}\right) = 1 \quad \text{and} \quad \lim_{T \rightarrow \infty} 0.5 e^{qT} = \infty. \quad \square$$

Problem 10: Use risk-neutral pricing to price a supershare, i.e., an option that pays $(\max(S(T) - K, 0))^2$ at the maturity of the option. In other words, compute

$$V(0) = e^{-rT} E_{RN}[(\max(S(T) - K, 0))^2],$$

where the expected value is computed with respect to the risk-neutral distribution of the price $S(T)$ of the underlying asset at maturity T , which is assumed to follow a lognormal process with drift r and volatility σ . Assume that the underlying asset pays no dividends, i.e., $q = 0$.

Solution: Recall that

$$S(T) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right)$$

and note that

$$S(T) \geq K \iff Z \geq \frac{\ln\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -d_2.$$

Then,

$$\begin{aligned} V(0) &= e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \left(S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right) - K\right)^2 e^{-\frac{x^2}{2}} dx \\ &= K^2 e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx \\ &\quad - 2KS(0) \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x - \frac{x^2}{2}\right) dx \\ &\quad + S^2(0) \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left((2r - \sigma^2)T + 2\sigma\sqrt{T}x - \frac{x^2}{2}\right) dx \quad (4.33) \end{aligned}$$

When pricing a plain vanilla call using risk-neutrality, we proved that

$$\begin{aligned} C_{BS}(0) &= S(0) \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x - \frac{x^2}{2}\right) dx \\ &\quad - Ke^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= S(0)N(d_1) - Ke^{-rT}N(d_2). \end{aligned}$$

In other words, we showed that

$$e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} dx = e^{-rT}N(d_2); \quad (4.34)$$

$$\frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x - \frac{x^2}{2}\right) dx = N(d_1) \quad (4.35)$$

From (4.33), (4.34), and (4.35), we conclude that

$$\begin{aligned} V(0) &= K^2e^{-rT}N(d_2) - 2KS(0)N(d_1) \\ &\quad + S^2(0) \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left((2r - \sigma^2)T + 2\sigma\sqrt{T}x - \frac{x^2}{2}\right) dx. \end{aligned} \quad (4.36)$$

The integral from (4.36) is computed by completing the square as follows:

$$\begin{aligned} &S^2(0) \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left((2r - \sigma^2)T + 2\sigma\sqrt{T}x - \frac{x^2}{2}\right) dx \\ &= S^2(0) \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left(-\frac{(x - 2\sigma\sqrt{T})^2}{2} + (2r + \sigma^2)T\right) dx \\ &= S^2(0)e^{(r+\sigma^2)T} \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} \exp\left(-\frac{(x - 2\sigma\sqrt{T})^2}{2}\right) dx \\ &= S^2(0)e^{(r+\sigma^2)T} \frac{1}{\sqrt{2\pi}} \int_{-(d_2+2\sigma\sqrt{T})}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \\ &= S^2(0)e^{(r+\sigma^2)T}N(d_2 + 2\sigma\sqrt{T}); \end{aligned} \quad (4.37)$$

we used the substitution $y = x - 2\sigma\sqrt{T}$ and the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a \exp\left(-\frac{y^2}{2}\right) dy = N(a).$$

From (4.36) and (4.37), we conclude that

$$V(0) = K^2e^{-rT}N(d_2) - 2KS(0)N(d_1) + S^2(0)e^{(r+\sigma^2)T}N(d_2 + 2\sigma\sqrt{T}). \quad \square$$

Problem 11: If the price of an asset follows a normal process, i.e., $dS = \mu dt + \sigma dX$, then

$$S(t_2) = S(t_1) + \mu(t_2 - t_1) + \sigma\sqrt{t_2 - t_1} Z, \quad \forall 0 < t_1 < t_2.$$

Assume that the risk free rate is constant and equal to r .

(i) Use risk neutrality to find the value of a call option with strike K and maturity T .

(ii) Use the Put–Call parity to find the value of a put option with strike K and maturity T , if the underlying asset follows a normal process as above.

Solution: (i) Using risk-neutral pricing, it follows that

$$C(0) = e^{-rT} E_{RN}[\max(S(T) - K, 0)],$$

where the expected value is computed with respect to $S(T)$ given by

$$S(T) = S(0) + rT + \sigma\sqrt{T} Z. \quad (4.38)$$

Note that

$$S(T) > K \quad \text{iff} \quad Z > \frac{K - S(0) - rT}{\sigma\sqrt{T}} = d.$$

Then,

$$\begin{aligned} C(0) &= e^{-rT} \frac{1}{\sqrt{2\pi}} \int_d^\infty (S(0) + rT + \sigma\sqrt{T}x - K) e^{-\frac{x^2}{2}} dx \\ &= (S(0) + rT)e^{-rT} \frac{1}{\sqrt{2\pi}} \int_d^\infty e^{-\frac{x^2}{2}} dx - Ke^{-rT} \frac{1}{\sqrt{2\pi}} \int_d^\infty e^{-\frac{x^2}{2}} dx \\ &\quad + \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}} \int_d^\infty xe^{-\frac{x^2}{2}} dx. \end{aligned}$$

Note that

$$\begin{aligned} \int_d^\infty xe^{-\frac{x^2}{2}} dx &= \lim_{t \rightarrow \infty} \int_d^t xe^{-\frac{x^2}{2}} dx = \lim_{t \rightarrow \infty} \left(-e^{-\frac{x^2}{2}}\right) \Big|_d^t = e^{-\frac{d^2}{2}}; \\ \frac{1}{\sqrt{2\pi}} \int_d^\infty e^{-\frac{x^2}{2}} dx &= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{x^2}{2}} dx = 1 - N(d) = N(-d), \end{aligned}$$

where $N(t)$ is the cumulative distribution of the standard normal variable.

We conclude that

$$C(0) = (S(0) + rT)e^{-rT}N(-d) - Ke^{-rT}N(-d) + \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}. \quad (4.39)$$

(ii) Regardless of the model used for describing the evolution of the price of the underlying asset, the Put–Call parity says that a portfolio made of a long position in a plain vanilla European call option and a short position in a plain vanilla European put option on the same asset and with the same strike and maturity as the call option has the same payoff at maturity as a long position in a unit of the underlying asset and a short cash position equal to the strike of the options. Using risk–neutral pricing, this can be written as

$$C(0) - P(0) = e^{-rT} E_{RN}[S(T) - K] = e^{-rT}(S(0) + rT - K), \quad (4.40)$$

since $E_{RN}[S(T)] = S(0) + rT$; cf. (4.38).

From (4.39) and (4.40), we obtain that

$$\begin{aligned} P(0) &= C(0) - (S(0) + rT)e^{-rT} + Ke^{-rT} \\ &= Ke^{-rT}(1 - N(-d)) - (S(0) + rT)e^{-rT}(1 - N(-d)) \\ &\quad + \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d^2}{2}} \\ &= Ke^{-rT}N(d) - (S(0) + rT)e^{-rT}N(d) + \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d^2}{2}}, \end{aligned}$$

since $1 - N(-d) = N(d)$. \square