

1.2 Supplemental Exercises

1. Compute

$$\int x^n \ln(x) dx.$$

2. Compute

$$\int x^n e^x dx.$$

3. Compute

$$\int (\ln(x))^n dx.$$

4. Show that

$$\left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}, \quad \forall x \geq 1.$$

5. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Assume that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which is uniformly bounded, i.e., there exists a constant C such that $|g(x)| \leq C$ for all $x \in \mathbb{R}$. Then, show that

$$\lim_{\sigma \searrow 0} \int_{-\infty}^{\infty} f(x)g(x) dx = g(\mu).$$

6. Let

$$g(y) = \sum_{i=1}^n \frac{c_i}{(1+y)^{t_i}}.$$

Compute $g'(y)$.

7. A derivative security pays a cash amount c if the spot price of the underlying asset at maturity is between K_1 and K_2 , where $0 < K_1 < K_2$, and expires worthless otherwise. How do you synthesize this derivative security (i.e., how do you recreate its payoff almost exactly) using plain vanilla call options?

8. Create a portfolio with the following payoff at time T :

$$V(T) = \begin{cases} 2S(T), & \text{if } 0 \leq S(T) < 20; \\ 60 - S(T), & \text{if } 20 \leq S(T) < 40; \\ S(T) - 20, & \text{if } 40 \leq S(T), \end{cases}$$

where $S(T)$ is the spot price at time T of a given asset. Use plain vanilla options with maturity T as well as cash positions and positions in the asset itself. Assume, for simplicity, that the asset does not pay dividends and that interest rates are zero.

9. Call options on the same underlying asset and with the same maturity, with strikes $K_1 < K_2 < K_3$, are trading for C_1 , C_2 and C_3 , respectively (no Bid–Ask spread), with $C_1 > C_2 > C_3$. Find necessary and sufficient conditions on the prices C_1 , C_2 and C_3 such that no–arbitrage exists corresponding to a portfolio made of positions in the three options.
10. Denote by C_{bid} and C_{ask} , and by P_{bid} and P_{ask} , respectively, the bid and ask prices for a plain vanilla European call and for a plain vanilla European put option, both with the same strike K and maturity T , and on the same underlying asset with spot price S and paying dividends continuously at rate q . Assume that the risk–free interest rates are constant equal to r . Find necessary and sufficient no–arbitrage conditions for C_{bid} , C_{ask} , P_{bid} , and P_{ask} .

1.3 Solutions to Supplemental Exercises

Problem 1: Compute

$$\int x^n \ln(x) \, dx.$$

Solution: If $n \neq -1$, we use integration by parts and find that

$$\begin{aligned} \int x^n \ln(x) \, dx &= \frac{x^{n+1}}{n+1} \ln(x) - \frac{1}{n+1} \int x^{n+1} \cdot \frac{1}{x} \, dx \\ &= \frac{x^{n+1} \ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C. \end{aligned}$$

For $n = -1$, we obtain that

$$\int \frac{\ln(x)}{x} \, dx = (\ln(x))^2 + C. \quad \square$$