2.2 Supplemental Exercises

1. Assume that the continuously compounded instantaneous rate curve $r(t)$ is given by
$$r(t) = \frac{0.05}{1 + \exp(-(1+t)^2)}.$$
(i) Use Simpson's Rule to compute the 1–year and 2–year discount factors with six decimal digits accuracy, and compute the 3–year discount factor with eight decimal digits accuracy.

   (ii) Find the value of a three year yearly coupon bond with coupon rate 5% (and face value 100).

2. Consider a six months plain vanilla European put option with strike 50 on a lognormally distributed underlying asset paying dividends continuously at 2%. Assume that interest rates are constant at 4%.

   Use risk–neutral valuation to write the value of the put as an integral over a finite interval. Find the value of the put option with six decimal digits accuracy using the Midpoint Rule and using Simpson's Rule. Also, compute the Black–Scholes value P_{BS} of the put and report the approximation errors of the numerical integration approximations at each step.

3. The prices of three call options with strikes 45, 50, and 55, on the same underlying asset and with the same maturity, are \$4, \$6, and \$9, respectively. Create a butterfly spread by going long a 45–call and a 55–call, and shorting two 50–calls. What are the payoff and the P&L at maturity of the butterfly spread? When would the butterfly spread be profitable? Assume, for simplicity, that interest rates are zero.

4. Dollar duration is defined as
$$D_{\$} = -\frac{\partial B}{\partial y}$$
and measures by how much the value of a bond portfolio changes for a small parallel shift in the yield curve.

   Similarly, dollar convexity is defined as
$$C_{\$} = \frac{\partial^2 B}{\partial y^2}.$$

Note that, unlike classical duration and convexity, which can only be computed for individual bonds, dollar duration and dollar convexity can be estimated for any bond portfolio, assuming all bond yields change by the same amount. In particular, for a bond with value B, duration D, and convexity C, the dollar duration and the dollar convexity can be computed as

$$D_{\$} = BD \quad \text{and} \quad C_{\$} = BC.$$

You invest \$1 million in a bond with duration 3.2 and convexity 16 and \$2.5 million in a bond with duration 4 and convexity 24.

(i) What are the dollar duration and dollar convexity of your portfolio?

(ii) If the yield goes up by ten basis points, find new approximate values for each of the bonds. What is the new value of the portfolio?

(iii) You can buy or sell two other bonds, one with duration 1.6 and convexity 12 and another one with duration 3.2 and convexity 20. What positions could you take in these bonds to immunize your portfolio (i.e., to obtain a portfolio with zero dollar duration and dollar convexity)?

2.3 Solutions to Supplemental Exercises

Problem 1: Assume that the continuously compounded instantaneous rate curve $r(t)$ is given by

$$r(t) \;=\; \frac{0.05}{1+\exp(-(1+t)^2)}.$$

Use Simpson's Rule to compute the 1–year and 2–year discount factors with six decimal digits accuracy, and compute the 3–year discount factor with eight decimal digits accuracy.

(ii) Find the value of a three year yearly coupon bond with coupon rate 5% (and face value 100).

Solution: (i) Recall that the discount factor corresponding to time t is

$$\exp\left(-\int_0^t r(\tau)\ d\tau\right).$$

Using Simpson's Rule, we obtain that the 1–year, 2–year, and 3–year discount factors are

$$\text{disc}(1) \;=\; 0.956595; \quad \text{disc}(2) \;=\; 0.910128; \quad \text{disc}(3) \;=\; 0.86574100.$$