3.2 Supplemental Exercises

1. What is the expected number of coin tosses of a fair coin in order to get two heads in a row? What if the coin is biased and the probability of getting heads is p?

2. What is the expected number of tosses in order to get k heads in a row for a biased coin with probability of getting heads equal to p?

3. Calculate the mean and variance of the uniform distribution on the interval [a,b].

4. Let X be a normally distributed random variable with mean μ and standard deviation $\sigma > 0$. Compute $E[\ |X|\ ]$ and $E[X^2]$.

5. Compute the expected value and variance of the Poisson distribution, i.e., of a random variable X taking only positive integer values with probabilities
$$P(X = k) \ = \ \frac{e^{-\lambda}\lambda^k}{k!}, \ \ \forall\ k \geq 0,$$
where $\lambda > 0$ is a fixed positive number.

6. Show that the values of a plain vanilla put option and of a plain vanilla call option with the same maturity and strike, and on the same underlying asset, are equal if and only if the strike is equal to the forward price.

7. You hold a portfolio made of a long position in 1000 put options with strike price 25 and maturity of six months, on a non–dividend–paying stock with lognormal distribution with volatility 30%, a long position in 400 shares of the same stock, which has spot price \$20, and \$10,000 in cash. Assume that the risk-free rate is constant at 4%.

   (i) How much is the portfolio worth?

   (ii) How do you adjust the stock position to make the portfolio Delta–neutral?

   (iii) A month later, the spot price of the underlying asset is \$24. What is new value of your portfolio, and how do you adjust the stock position to make the portfolio Delta–neutral?

8. You hold a portfolio with $\Delta(\Pi) = 300$, $\Gamma(\Pi) = 100$, and $\text{vega}(\Pi) = 89$. You can trade in the underlying asset, in a call option with

$$\Delta(C) = 0.2; \quad \Gamma(C) = 0.1; \quad \text{vega}(C) = 0.1,$$

and in a put option with

$$\Delta(P) = -0.8; \quad \Gamma(P) = 0.3; \quad \text{vega}(P) = 0.2.$$

What trades do you make to obtain a Δ–, Γ–, and vega–neutral portfolio?

3.3 Solutions to Supplemental Exercises

Problem 1: What is the expected number of coin tosses of a fair coin in order to get two heads in a row? What if the coin is biased and the probability of getting heads is p?

Solution: If p is the probability of the coin toss resulting in heads, then the probability of the coin toss resulting in tails is $1 - p$.

The outcomes of the first two tosses are as follows:

• If the first toss is tails, which happens with probability $1 - p$, then the process resets and the expected number of tosses increases by 1.

• If the first toss is heads, and if the second toss is also heads, which happens with probability p^2, then two consecutive heads were obtained after two tosses.

• If the first toss is heads, and if the second toss is tails, which happens with probability $p(1 - p)$, then the process resets and the expected number of tosses increases by 2.

If $E[X]$ denotes the expected number of tosses in order to get two heads in a row, we conclude that

$$E[X] \;=\; (1-p)(1+E[X]) \;+\; 2p^2 + p(1-p)(2+E[X]). \tag{3.20}$$

We solve (3.20) for $E[X]$ and obtain that

$$E[X] \;=\; \frac{1+p}{p^2}.$$

For an unbiased coin, i.e., for $p = \frac{1}{2}$, we find that $E[X] = 6$, and therefore the expected number of coin tosses to obtain two heads in a row is 6. □