5.2 Supplemental Exercises

1. (i) Let $g(x)$ be an infinitely differentiable function. Find the linear and quadratic Taylor approximations of $e^{g(x)}$ around the point 0.

   (ii) Use the result above to compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$.

   (iii) Compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$ by using Taylor approximations of e^x and e^{x^2}.

2. Show that
$$e^{-x} - \frac{1}{1+x} \ = \ O(x^2), \quad \text{as} \ \ x \to 0.$$

3. Compute the Taylor series expansion of
$$\ln\left(\frac{1+x}{1-x}\right)$$
   around the point 0, and find its radius of convergence.

4. Recall that
$$\left(1+\frac{1}{x}\right)^x \ < \ e \ < \ \left(1+\frac{1}{x}\right)^{x+1}, \quad \forall\, x \geq 1.$$
   Prove that
$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}-\frac{1}{12x}} \ < \ e \ < \ \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}, \quad \forall\, x \geq 1.$$

5. (i) Find the radius of convergence of the series
$$1 + \frac{x^4}{2!} + \frac{x^8}{4!} + \frac{x^{12}}{6!} + \ldots \tag{5.25}$$

   (ii) Show that the series from (5.25) is the Taylor series expansion of the function
$$\frac{e^{x^2} + e^{-x^2}}{2}.$$

6. The goal of this exercise is to compute

$$\int_0^1 \ln(1-x)\ln(x)\ dx. \tag{5.26}$$

(i) Show that

$$\lim_{x \searrow 0} (\ln(1-x)\ln(x)) \;=\; \lim_{x \nearrow 1} (\ln(1-x)\ln(x)) \;=\; 0,$$

and conclude that the integral (5.26) can be regarded as a definite integral.

(ii) Use the Taylor series expansion of $\ln(1-x)$ for $|x| < 1$ to show that

$$\int_0^1 \ln(1-x)\ln(x)\ dx \;=\; -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln(x)\ dx.$$

(iii) Prove that

$$\int_0^1 \ln(1-x)\ln(x)\ dx \;=\; \sum_{k=1}^{\infty} \frac{1}{n(n+1)^2}.$$

(iv) Use that fact that

$$\sum_{k=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

to obtain that

$$\int_0^1 \ln(1-x)\ln(x)\ dx \;=\; 2 - \frac{\pi^2}{6}.$$

7. Consider an ATM put option with strike 40 on an asset with volatility 30% and paying 2% dividends continuously. Assume that the interest rates are constant at 4.5%. Compute the relative approximation error to the Black–Scholes value of the option of the approximate value

$$P_{approx, r \neq 0, q \neq 0} \;=\; \sigma S \sqrt{\frac{T}{2\pi}} \left(1 - \frac{(r+q)T}{2}\right) \;-\; \frac{(r-q)T}{2} S,$$

if the put option expires in 1, 3, 5, 10, and 20 years.