6.2 Supplemental Exercises

1. Let $f(x) = x^3e^x - 6e^x$. Show that the central difference approximation for $f'(x)$ around the point 0 is a fourth order approximation.

2. Let
$$f(S) \;=\; \frac{C(K+x,T) - 2C(K,T) + C(K-x,T)}{x^2},$$
where, e.g., $C(K,T)$ denotes the value of a plain vanilla call option with strike K and maturity T on an underlying asset with spot price S following a lognormal distribution. Show that, for any continuous function $g : \mathbb{R} \to \mathbb{R}$,
$$\lim_{x \searrow 0} \int_{-\infty}^{\infty} f(S)g(S)\, dS \;=\; g(K).$$

3. (i) Show that the approximate formula
$$1 \;+\; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma}{\Theta} \;\approx\; 0$$
connecting the Γ and the Θ of plain vanilla European options is exact if the underlying asset pays no dividends and if the risk–free interest rates are zero. In other words, for, e.g., call options, show that, if $r = q = 0$, then
$$1 \;+\; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)} \;=\; 0.$$
(ii) If $q = 0$ but $r \neq 0$, show that
$$1 \;+\; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)} \;=\; \frac{1}{1 + \frac{\sigma}{2r\sqrt{T-t}} \frac{N'(d_2)}{N(d_2)}}.$$
(iii) Consider a six months plain vanilla European call option on an underlying asset with spot price 50 and volatility 30%. Assume that the interest rates are constant at 4%. If the asset pays no dividends, compute
$$1 \;+\; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)}$$
if the options are at–the–money, 10%, 20%, 30%, and 50% in–the–money, and 10%, 20%, 30%, and 50% out–of–the–money, respectively.

What happens if the asset pays dividends continuously at a 3% rate?

4. Consider a six months 5% in–the–money plain vanilla European call option with strike 30 on an underlying asset with volatility 20%, paying dividends continuously at a 2% rate. Assume that the interest rates are constant at 5%.

   (i) Use central differences to compute the finite difference approximations Δ_c and Γ_c for Δ and Γ, respectively, i.e.,

$$\begin{aligned}\Delta_c &= \frac{C(S+dS)-C(S-dS)}{2dS};\\ \Gamma_c &= \frac{C(S+dS)-2C(S)+C(S-dS)}{(dS)^2},\end{aligned}$$

   for $dS = 10^{-i}$ with $i = 1 : 12$, where, e.g., $C(S+dS) = C(S+dS, K, T, \sigma, r)$ denotes the Black–Scholes value of the call option corresponding to a spot price $S + dS$ of the underlying asset.

   (ii) Compute the Delta and Gamma of the call using the Black–Scholes formula, and the approximation errors $|\Delta_c - \Delta|$ and $|\Gamma_c - \Gamma|$. Note that these approximation errors stop improving, or even worsen, as dS becomes too small. How do you explain this?

6.3 Solutions to Supplemental Exercises

Problem 1: Let $f(x) = x^3e^x - 6e^x$. Show that the central difference approximation for $f'(x)$ around the point 0 is a fourth order approximation.

Solution: Recall that, in general, the central difference approximation of the first derivative is a second order approximation, i.e.,

$$f'(0) = \frac{f(h)-f(-h)}{2h} + O\left(h^2\right), \quad \text{as } h \to 0. \tag{6.28}$$

To see why, for the function $f(x) = x^3e^x - 6e^x$, the central difference approximation for $f'(x)$ around the point 0 is a fourth order approximation, we investigate how the approximation (6.28) is derived.

The Taylor approximation of $f(x)$ around the point 0 for $n = 5$ is

$$\begin{aligned}f(x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f^{(3)}(0) + \frac{x^4}{24}f^{(4)}(0) + \frac{x^5}{120}f^{(5)}(0)\\ &\quad + O\left(x^6\right), \quad \text{as } x \to 0.\end{aligned} \tag{6.29}$$

We let $x = h$ and $x = -h$ in (6.29) and sum up the two resulting formulas. After solving for $f'(0)$ we obtain

$$f'(0) \ = \ \frac{f(h) - f(-h)}{2h} \ - \ \frac{h^2}{6} f^{(3)}(0) \ - \ \frac{h^4}{120} f^{(5)}(0) \ + \ O\left(h^5\right), \qquad (6.30)$$

as $h \to 0$.

For $f(x) = x^3 e^x - 6e^x$, we find that $f^{(3)}(x) = (x^3 + 9x^2 + 18x)e^x$, and thus that $f^{(3)}(0) = 0$. Also, $f^{(5)}(0) = 54 \neq 0$ and (6.30) becomes

$$f'(0) \ = \ \frac{f(h) - f(-h)}{2h} \ - \ \frac{9h^4}{20} \ + \ O\left(h^5\right) \ = \ \frac{f(h) - f(-h)}{2h} \ + \ O\left(h^4\right),$$

as $h \to 0$. In other words, the central difference approximation for $f'(x)$ around the point 0 is a fourth order approximation. □

Problem 2: Let

$$f(S) \ = \ \frac{C(K+x,T) - 2C(K,T) + C(K-x,T)}{x^2},$$

where, e.g., $C(K,T)$ denotes the value of a plain vanilla call option with strike K and maturity T on an underlying asset with spot price S following a lognormal distribution. Show that, for any continuous function $g : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \searrow 0} \int_{-\infty}^{\infty} f(S) g(S) \, dS \ = \ g(K). \qquad (6.31)$$

Solution: From the definition of $f(S)$, it is easy to see that

$$\begin{aligned} f(S) \ &= \ \frac{1}{x^2} \left(\max(S - (K-x), 0) - 2\max(S-K, 0) + \max(S - (K+x), 0) \right) \\ &= \ \begin{cases} 0, & \text{if } 0 < S \le K - x; \\ \frac{S-(K-x)}{x^2}, & \text{if } K - x < S \le K; \\ \frac{K+x-S}{x^2}, & \text{if } K < S \le K + x; \\ 0, & \text{if } K + x < S. \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} \int_{-\infty}^{\infty} f(S) g(S) dS \ &= \ \int_{K-x}^{K} \frac{S - (K-x)}{x^2} g(S) dS + \int_{K}^{K+x} \frac{K + x - S}{x^2} g(S) dS \\ &= \ \frac{1}{x^2} \int_0^x z g(K - x + z) dz + \frac{1}{x^2} \int_0^x w g(K + x - w) dw, \end{aligned}$$

where we used the substitutions $z = S - (K - x)$ and $w = K + x - S$ for the two integrals above, respectively.

Let $z = xy$ and $w = xt$. Then $dz = x\ dy$, $dw = x\ dt$, and we find that

$$\int_{-\infty}^{\infty} f(S)g(S)\ dS \ = \ \int_0^1 yg(K-x+xy)\ dy \ + \ \int_0^1 tg(K+x-xt)\ dt. \quad (6.32)$$

We let $x \searrow 0$ in (6.32). Since the function $g : \mathbb{R} \to \mathbb{R}$ is continuous, we obtain that

$$\lim_{x\searrow 0} \int_{-\infty}^{\infty} f(S)g(S)\ dS \ = \ g(K)\int_0^1 y\ dy \ + \ g(K)\int_0^1 t\ dt \ = \ g(K), \quad (6.33)$$

which is what we wanted to show; cf. (6.31).

For the sake of completeness, we provide rigorous proof of the fact that (6.32) becomes (6.33) when $x \searrow 0$. To do so, it is enough to show that

$$\lim_{x\searrow 0} \int_0^1 yg(K - x + xy)\ dy \ = \ g(K)\int_0^1 y\ dy.$$

Let $\epsilon > 0$ arbitrary. Since g is continuous, it follows that there exists $\delta > 0$ such that $|g(K) - g(\tau)| < \epsilon$ for all τ such that $|K - \tau| < \delta$. Let $x \in (0, \delta)$ and $y \in (0, 1)$. Then $|K - (K - x + xy)| = x(1 - y) < \delta$ and therefore

$$|g(K) - g(K - x + xy)| < \epsilon, \ \ \forall\ 0 < x < \delta,\ 0 < y < 1.$$

Therefore, it is easy to see that, for any $0 < x < \delta$,

$$\begin{aligned} \left| \int_0^1 yg(K - x + xy)dy - g(K)\int_0^1 ydy \right| \ &\le \ \int_0^1 y|g(K) - g(K - x + xy)|dy \\ &< \ \epsilon \int_0^1 y\ dy \ = \ \frac{\epsilon}{2}. \end{aligned}$$

In other words, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_0^1 yg(K - x + xy)\ dy \ - \ g(K)\int_0^1 y\ dy \right| \ < \ \frac{\epsilon}{2}, \ \ \forall\ 0 < x < \delta.$$

Then, by definition,

$$\lim_{x\searrow 0} \left| \int_0^1 yg(K - x + xy)\ dy \ - \ g(K)\int_0^1 y\ dy \right| \ = \ 0. \quad \square$$

Problem 3: (i) Show that the approximate formula

$$1 \ + \ \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma}{\Theta} \ \approx \ 0$$

connecting the Γ and the Θ of plain vanilla European options is exact if the underlying asset pays no dividends and if the risk–free interest rates are zero. In other words, for, e.g., call options[1],

$$1 \; + \; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)} \;=\; 0.$$

(ii) If $q = 0$ and $r \neq 0$, show that

$$1 \; + \; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)} \;=\; \frac{1}{1 + \frac{\sigma}{2r\sqrt{T}} \frac{N'(d_2)}{N(d_2)}}.$$

(iii) Consider a six months plain vanilla European call option on an underlying asset with spot price 50 and volatility 30%. Assume that the interest rates are constant at 4%. If the asset pays no dividends, compute

$$1 \; + \; \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)}$$

if the options are at–the–money, 10%, 20%, 30%, and 50% in–the–money, and 10%, 20%, 30%, and 50% out–of–the–money, respectively.

What happens if the asset pays dividends continuously at a 3% rate?

Solution: Recall that the Γ and the Θ of a plain vanilla European option are

$$\Gamma(C) \;=\; \frac{e^{-qT}}{\sigma S \sqrt{2\pi T}} e^{-\frac{d_1^2}{2}}; \qquad (6.34)$$

$$\Theta(C) \;=\; -\frac{\sigma S e^{-qT}}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}} \;+\; qSe^{-qT} N(d_1) \;-\; rKe^{-rT} N(d_2), \qquad (6.35)$$

where $d_1 = \left(\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T\right) / \left(\sigma\sqrt{T}\right)$ and $d_2 = d_1 - \sigma\sqrt{T}$.

(i) For $r = q = 0$, we obtain from (6.34) and (6.35) that

$$\Gamma(C) \;=\; \frac{1}{\sigma S \sqrt{2\pi T}} e^{-\frac{d_1^2}{2}}; \quad \Theta(C) \;=\; -\frac{\sigma S}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}}.$$

Then,

$$1 + \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma(C)}{\Theta(C)} \;=\; 1 + \frac{\sigma^2 S^2}{2} \left(-\frac{2}{\sigma^2 S^2}\right) \;=\; 0.$$

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[1] Note that, if $r = q = 0$, then $\Gamma(P) = \Gamma(C)$ and $\Theta(P) = \Theta(C)$.

(ii) For $q = 0$, we obtain from (6.34) and (6.35) that

$$\begin{aligned}\Gamma(C) &= \frac{1}{\sigma S\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}};\\ \Theta(C) &= -\frac{\sigma S}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}} \;-\; rKe^{-rT}N(d_2).\end{aligned}$$

Then,

$$\begin{aligned}1 + \frac{\sigma^2 S^2}{2}\cdot\frac{\Gamma(C)}{\Theta(C)} &= 1 - \frac{\frac{\sigma S}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}}}{\frac{\sigma S}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}} \;+\; rKe^{-rT}N(d_2)}\\ &= \frac{rKe^{-rT}N(d_2)}{\frac{\sigma S}{2\sqrt{2\pi T}} e^{-\frac{d_1^2}{2}} \;+\; rKe^{-rT}N(d_2)}\\ &= \frac{1}{1 + \frac{\sigma}{2r\sqrt{T}}\cdot\frac{S}{Ke^{-rT}N(d_2)}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}}\\ &= \frac{1}{1 + \frac{\sigma}{2r\sqrt{T}}\cdot\frac{SN'(d_1)}{Ke^{-rT}N(d_2)}}. \qquad (6.36)\end{aligned}$$

since $N'(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ for all $t \in \mathbb{R}$.

Recall that the “magic” of Greek computations is due to the following result:

$$S\; N'(d_1) \;=\; Ke^{-rT}\; N'(d_2);$$

cf. Lemma 3.15 of [2] for $q = 0$. Then, (6.36) becomes

$$1 + \frac{\sigma^2 S^2}{2}\cdot\frac{\Gamma(C)}{\Theta(C)} \;=\; \frac{1}{1 + \frac{\sigma}{2r\sqrt{T}}\frac{N'(d_2)}{N(d_2)}}.$$

(iii) Let $S = 50$, $T = 0.5$, $\sigma = 0.3$, and $r = 0.04$. The table below records the values of

$$1 \;+\; \frac{\sigma^2 S^2}{2}\cdot\frac{\Gamma(C)}{\Theta(C)}$$

(denoted by “Value”) both for $q = 0$, and for $q = 0.03$, for the following values of the moneyness of the option:

$$\frac{S}{K} \;=\; \{1, 1.1, 1.2, 1.3, 1.5, 0.9, 0.8, 0.7, 0.5\},$$

corresponding to call options that are at–the–money, 10%, 20%, 30%, and 50% in–the–money, and 10%, 20%, 30%, and 50% out–of–the–money, respectively:

S/K	Value for $q = 0$	Value for $q = 0.03$
1	0.1897	0.0238
1.1	0.2582	0.0233
1.2	0.3518	0.0144
1.3	0.4711	-0.0187
1.5	0.7361	-0.5503
0.9	0.1412	0.0214
0.8	0.1068	0.0184
0.7	0.0822	0.0155
0.5	0.0505	0.0107

We note that the approximation

$$1 + \frac{\sigma^2 S^2}{2} \cdot \frac{\Gamma}{\Theta} \approx 0$$

is better for deep out–of–the–money options (corresponding to small values of S/K) and is worse for deep in–the–money options (corresponding to large values of S/K). Also, for this particular case, the approximation is more accurate if the underlying asset pays dividends. □

Problem 4: Consider a six months 5% in–the–money plain vanilla European call option with strike 30 on an underlying asset with spot price 20 and volatility 20%, paying dividends continuously at a 2% rate. Assume that the interest rates are constant at 5%.

(i) Use central differences to compute the finite difference approximations Δ_c and Γ_c for Δ and Γ, respectively, i.e.,

$$\begin{aligned}\Delta_c &= \frac{C(S+dS) - C(S-dS)}{2dS};\\ \Gamma_c &= \frac{C(S+dS) - 2C(S) + C(S-dS)}{(dS)^2},\end{aligned}$$

for $dS = 10^{-i}$ with $i = 1:12$, where, e.g., $C(S+dS) = C(S+dS, K, T, \sigma, r)$ denotes the Black–Scholes value of the call option corresponding to a spot price $S + dS$ of the underlying asset.

(ii) Compute the Delta and Gamma of the call using the Black–Scholes formula, and the approximation errors $|\Delta_c - \Delta|$ and $|\Gamma_c - \Gamma|$. Note that these approximation errors stop improving, or even worsen, as dS becomes too small. How do you explain this?

Solution: The spot price $S = 31.5$ corresponds to a 5% ITM call with $K = 30$. We find that $\Delta = 0.692130579727$ and $\Gamma = 0.077379043990$.

The central finite difference approximations Δ_c and the approximation errors $|\Delta_c - \Delta|$ are recorded in the table below:

dS	Δ_c	$\lvert\Delta_c - \Delta\rvert$
0.1	0.692112731743	0.000017847983
0.01	0.692131730564	0.000001150838
0.001	0.692131920566	0.000001340839
0.0001	0.692131922513	0.000001342786
10^{-5}	0.692131922087	0.000001342360
10^{-6}	0.692131918000	0.000001338274
10^{-7}	0.692131916226	0.000001336498
10^{-8}	0.692131862934	0.000001283207
10^{-9}	0.692132573477	0.000001993750
10^{-10}	0.692104151767	0.000026427960
10^{-11}	0.691890988946	0.000239590780
10^{-12}	0.687450096848	0.004680482879

The approximations became more precise when dS decreased, until $dS = 10^{-8}$; the best approximation was within about 10^{-6} of Δ. However, for values of dS smaller than 10^{-9}, the finite difference approximations deteriorated very quickly.

To explain this phenomenon, denote the exact value[2] of Delta by Δ_{exact}. Note that the value of Δ is given by the Black–Scholes formula, i.e.,

$$\Delta \;=\; \Delta_{BS} \;=\; e^{-qT}N(d_1).$$

This value is computed using a numerical approximation of $N(d_1)$ that is accurate within $7.5 \cdot 10^{-7}$; cf. [1], page 932. In other words, we only know that

$$|\Delta_{BS} - \Delta_{exact}| \;<\; 10^{-6}. \tag{6.37}$$

When computing the finite difference approximation Δ_c, we use a numerical estimation of the Black–Scholes formula to compute $C(S + dS)$ and $C(S - dS)$ which once again involves the numerical approximation of the cumulative density of the standard normal variable. In other words,

$$\Delta_c \;=\; \frac{C_{BS}(S + dS) - C_{BS}(S - dS)}{2dS}. \tag{6.38}$$

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[2] Note that Δ_{exact} is a theoretical value, and is not the Δ from the table above.

Denote by $C_{exact}(S+dS)$ and $C_{exact}(S-dS)$ the exact values of the options. Since the central finite difference is a second order approximation, it follows that, for *exact* values of Delta and of the call options,

$$\Delta_{exact} \;=\; \frac{C_{exact}(S+dS) - C_{exact}(S-dS)}{2dS} \;+\; O\left((dS)^2\right), \qquad (6.39)$$

as $dS \to 0$.

Since $C_{BS}(S) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$, and since $N(d_1)$ and $N(d_2)$ are computed numerically within 10^{-6} of their exact value, it follows that

$$\begin{aligned} |C_{BS}(S+dS) - C_{exact}(S+dS)| &< \alpha 10^{-6}; && (6.40)\\ |C_{BS}(S-dS) - C_{exact}(S-dS)| &< \alpha 10^{-6}, && (6.41) \end{aligned}$$

where α is a constant proportional to the values of S and K,

Using (6.38) and(6.39) we find that

$$\begin{aligned} \Delta_c - \Delta_{BS} &= (\Delta_c - \Delta_{exact}) + (\Delta_{exact} - \Delta_{BS})\\ &= \frac{C_{BS}(S+dS) - C_{exact}(S+dS)}{2dS}\\ &\quad - \frac{C_{BS}(S-dS) - C_{exact}(S-dS)}{2dS}\\ &\quad + \Delta_{exact} - \Delta_{BS} \;+\; O\left((dS)^2\right), \qquad (6.42) \end{aligned}$$

as $dS \to 0$.

The only estimate we can find using (6.37), (6.40), (6.41), and (6.42) for the approximation of Δ_{BS} by Δ_c as $dS \to 0$ is

$$\begin{aligned} |\Delta_c - \Delta_{BS}| &\le \frac{|C_{BS}(S+dS) - C_{exact}(S+dS)|}{2dS}\\ &\quad + \frac{|C_{BS}(S-dS) - C_{exact}(S-dS)|}{2dS}\\ &\quad + |\Delta_{exact} - \Delta_{BS}| \;+\; O\left((dS)^2\right)\\ &\le \frac{\alpha 10^{-6}}{dS} \;+\; 10^{-6} \;+\; O\left((dS)^2\right), \qquad (6.43) \end{aligned}$$

as $dS \to 0$.

While the approximation error $|\Delta_c - \Delta|$ may be better in practice, the bound (6.43) provides the intuition behind the fact that, for dS too small, the numerical approximation error $|\Delta_c - \Delta| = |\Delta_c - \Delta_{BS}|$ deteriorates as $\frac{\alpha 10^{-6}}{dS}$ becomes large.

The central finite difference approximations Γ_c and the approximation errors $|\Gamma_c - \Gamma|$ are recorded in the table below:

dS	Γ_c	$\lvert\Gamma_c - \Gamma\rvert$
0.1	0.077370009586	0.000009034404
0.01	0.077371495486	0.000007548504
0.001	0.077371502982	0.000007541008
0.0001	0.077371709040	0.000007334951
10^{-5}	0.077271522514	0.000107521476
10^{-6}	0.074606987255	0.002772056735
10^{-7}	-0.355271367880	0.432650411870
10^{-8}	71.054273576010	70.976894532020

For $dS \leq 10^{-9}$, the values of Γ_c increased dramatically, reaching 10^9 for $dS = 10^{-12}$, and were no longer recorded. The finite difference approximations of Γ became more precise while dS decreased to 10^{-4}, but were much worse after that; the best approximation was within 10^{-5} of Γ. The reason for this is similar to the one explained above for the finite difference approximations of Δ. $\square$