7.2 Supplemental Exercises

1. Compute

$$\int\int_D x \, dxdy,$$

where

$$D \;=\; \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, \; 1 \leq xy \leq 2, \; 1 \leq \frac{y}{x} \leq 2\}.$$

Hint: The change of variables $s = xy$ and $t = \frac{y}{x}$ maps the domain D into the rectangle $[1,2] \times [1,2]$.

2. Which number is larger, e^π or π^e?

3. Let $u, v : [0, \infty) \to [0, \infty)$ be two continuous functions with positive values. Assume that there exists a constant $M > 0$ such that

$$u(x) \;\leq\; M + \int_0^x u(t)v(t) \, dt, \quad \forall \, x \geq 0.$$

Show that

$$u(x) \;\leq\; M \exp\left(\int_0^x v(t) \, dt\right), \quad \forall \, x \geq 0.$$

Hint: Investigate the monotonicity of the function

$$\left(M + \int_0^x u(t)v(t) \, dt\right) \exp\left(-\int_0^x v(t) \, dt\right).$$

Note: This is a version of Gronwall's inequality, and it is needed, e.g., to prove the uniqueness of the solution of an initial value problem for ordinary differential equations.

4. What does the boundary condition $V(B,t) = R$ for a down–and–out call with barrier B and rebate $R > 0$ correspond to for the function $u(x,\tau)$ defined as follows: $V(S,t) = \exp(-ax - b\tau)u(x,\tau)$, where

$$x = \ln\left(\frac{S}{K}\right), \quad \tau = \frac{(T-t)\sigma^2}{2}, \quad a = \frac{r-q}{\sigma^2} - \frac{1}{2}, \quad b = \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}.$$

5. Assume that the function $V(S, I, t)$ satisfies the PDE

$$\frac{\partial V}{\partial t} + \ln S \frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \qquad (7.21)$$

Consider the following change of variables:

$$V(S, I, t) = F(y, t), \quad \text{where} \quad y = \frac{I + (T-t)\ln S}{T}.$$

Show that $F(y, t)$ satisfies the following PDE:

$$\frac{\partial F}{\partial t} + \frac{\sigma^2 (T-t)^2}{2T^2}\frac{\partial^2 F}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{T-t}{T}\frac{\partial F}{\partial y} - rF = 0.$$

Note: The values of Asian options with continuously sampled geometric average satisfy the PDE (7.21).

6. One way to see that American calls on non–dividend–paying assets are never optimal to exercise is to note that the Black–Scholes value of the European call is always greater than the intrinsic premium $S - K$, for $S \geq K$.

Show that this argument does not work for dividend–paying assets. In other words, prove that the Black–Scholes value of the European call is smaller than $S - K$ for S large enough, if the underlying asset pays dividends continuously at the rate $q > 0$ (and regardless of how small q is).

7. For the same maturity, options with different strikes are traded simultaneously. The goal of this problem is to compute the rate of change of the implied volatility as a function of the strike of the options.

In other words, assume that S, T, q and r are given, and let $C(K)$ be the (known) value of a call option with maturity T and strike K. Assume that options with all strikes K exist. Define the implied volatility $\sigma_{imp}(K)$ as the unique solution to

$$C(K) = C_{BS}(K, \sigma_{imp}(K)),$$

where $C_{BS}(K, \sigma_{imp}(K)) = C_{BS}(S, K, T, \sigma_{imp}(K), r, q)$ represents the Black–Scholes value of a call option with strike K on an underlying asset following a lognormal model with volatility $\sigma_{imp}(K)$. Find an implicit differential equation satisfied by $\sigma_{imp}(K)$, i.e., find

$$\frac{\partial \sigma_{imp}(K)}{\partial K}$$

as a function of $\sigma_{imp}(K)$.