is fully deterministic. To avoid arbitrage opportunities, the value of this portfolio must grow at the same rate as the risk free interest rate r, i.e.,

$$d\Pi \;=\; r\Pi \; dt. \tag{5.64}$$

From (5.63) and (5.64), we conclude that

$$r\Pi \;\approx\; \Theta \;+\; \frac{\sigma^2 S^2}{2}\Gamma. \tag{5.65}$$

We note that the approximate formula (5.65) is actually an equality for plain vanilla European options; cf. (6.53) for $q = 0$. This equality can be derived using the Black–Scholes PDE; see section 6.4.2 for details.

5.5 Approximation of the Black–Scholes formula for at–the–money options

For at-the-money options, approximations to the Black–Scholes formula that do not require estimating the cumulative density function of the standard normal distributions can be derived. These formulas are easy to compute and, in many cases, very accurate.

In section 5.5.1, we present several such approximations formulas, including approximations for the Greeks and for the implied volatility. These formulas are proved in section 5.5.2. In section 5.5.3, we derive a theoretical bound for the precision of the approximation of the price for ATM options on non–dividend–paying assets for the case of zero interest rates. We also investigate the relative approximation error of the formulas from section 5.5.1 for two particular examples.

5.5.1 Several ATM approximations formulas

The most commonly used approximation formulas are for at-the-money options on assets paying no dividends, with zero risk-free rates:

$$\boxed{r = 0;\; q = 0: \quad P \;=\; C \;\approx\; \sigma S\sqrt{\tfrac{T}{2\pi}} \;\approx\; 0.4\,\sigma S\sqrt{T}}$$

For the general case when r and q are nonzero, the following approximation formulas can be used for pricing ATM plain vanilla European options:

$$r \neq 0;\; q \neq 0: \quad C \;\approx\; \sigma S\sqrt{\tfrac{T}{2\pi}}\left(1 - \tfrac{(r+q)T}{2}\right) \;+\; \tfrac{(r-q)T}{2}S$$

$$r \neq 0;\; q \neq 0: \quad P \;\approx\; \sigma S\sqrt{\tfrac{T}{2\pi}}\left(1 - \tfrac{(r+q)T}{2}\right) \;-\; \tfrac{(r-q)T}{2}S$$

Approximation formulas can also be obtained for the implied volatility of ATM options:

$$r = q = 0: \ \sigma_{imp} \approx \frac{C\sqrt{2\pi}}{S\sqrt{T}} \approx \frac{2.5\ C}{S\sqrt{T}}$$

$$r \neq 0;\ q \neq 0: \ \sigma_{imp} \approx \frac{\sqrt{2\pi}}{S\sqrt{T}} \frac{C - \frac{(r-q)T}{2}S}{1-\frac{(r+q)T}{2}} \approx \frac{\sqrt{2\pi}}{S\sqrt{T}} \frac{P + \frac{(r-q)T}{2}S}{1-\frac{(r+q)T}{2}}$$

If $r = q = 0$, the following approximation formulas can be used for the Greeks of ATM options (similar, but more complicated formulas exist for $r, q \neq 0$):

$$\Delta(C) \approx \frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} \approx \frac{1}{2} + 0.2\sigma\sqrt{T}$$

$$\Delta(P) \approx -\frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} \approx -\frac{1}{2} + 0.2\sigma\sqrt{T}$$

$$\Gamma(C) = \Gamma(P) \approx \frac{1}{\sigma S\sqrt{2\pi T}}\left(1 - \frac{\sigma^2 T}{8}\right) \approx \frac{0.4}{\sigma S\sqrt{T}}\left(1 - \frac{\sigma^2 T}{8}\right)$$

$$\mathrm{vega}(C) = \mathrm{vega}(P) \approx S\sqrt{\frac{T}{2\pi}}\left(1 - \frac{\sigma^2 T}{8}\right) \approx 0.4S\sqrt{T}\left(1 - \frac{\sigma^2 T}{8}\right)$$

$$\Theta(C) = \Theta(P) \approx -\frac{\sigma S}{2\sqrt{2\pi T}}\left(1 - \frac{\sigma^2 T}{8}\right) \approx -\frac{0.2\sigma S}{\sqrt{T}}\left(1 - \frac{\sigma^2 T}{8}\right)$$

$$\rho(C) \approx ST\left(\frac{1}{2} - \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}}\right) \approx ST\left(\frac{1}{2} - 0.2\sigma\sqrt{T}\right)$$

$$\rho(P) \approx -ST\left(\frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}}\right) \approx -ST\left(\frac{1}{2} + 0.2\sigma\sqrt{T}\right)$$

5.5.2 Deriving the ATM approximations formulas

We begin by proving that, if $r = 0$ and $q = 0$, the prices of the ATM call and ATM put are equal, and can be approximated as

$$P = C \approx \sigma S\sqrt{\frac{T}{2\pi}}, \tag{5.66}$$

or, in a simpler to compute but less precise way, as

$$P = C \approx 0.4\,\sigma S\sqrt{T}, \tag{5.67}$$

since $\frac{1}{\sqrt{2\pi}} = 0.39894228 \approx 0.4.$

Since $r = q = 0$ and $K = S$, it is easy to see from the Put–Call parity (1.47) that the values of the put option and call options are the same:

$$P \;=\; Ke^{-rT} - S + C \;=\; S - S + C \;=\; C. \tag{5.68}$$

For an ATM call and assuming zero interest rates and a non-dividend-paying asset, i.e., for $K = S$, $r = 0$, and $q = 0$, the Black–Scholes formula (3.53) corresponding to $t = 0$ can be written as

$$C \;=\; SN\left(\frac{\sigma\sqrt{T}}{2}\right) \;-\; SN\left(-\frac{\sigma\sqrt{T}}{2}\right). \tag{5.69}$$

From (3.43), we find that $N\left(-\frac{\sigma\sqrt{T}}{2}\right) = 1 - N\left(\frac{\sigma\sqrt{T}}{2}\right)$ and (5.69) becomes

$$C \;=\; 2S\;\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2}\right). \tag{5.70}$$

We approximate the term $N\left(\frac{\sigma\sqrt{T}}{2}\right)$ using a Taylor approximation around 0. From (5.7) for $f(x) = N(x)$, $a = 0$, and $n = 2$, we obtain that

$$N(x) \;=\; N(0) \;+\; xN'(0) \;+\; \frac{x^2}{2}N''(0) \;+\; O(x^3), \tag{5.71}$$

as $x \to 0$. Recall that

$$N(t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{y^2}{2}}\,dy.$$

It is easy to see that $N(0) = \frac{1}{2}$; cf. (3.43) for $a = 0$. From Lemma 2.3, we obtain that $N'(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$, and therefore $N''(t) = -\frac{1}{\sqrt{2\pi}}te^{-\frac{t^2}{2}}$. Thus, $N(0) = \frac{1}{2}$, $N'(0) = \frac{1}{\sqrt{2\pi}}$, $N''(0) = 0$, and the Taylor expansion (5.71) of $N(x)$ around the point 0 becomes[4]

$$N(x) \;=\; \frac{1}{2} \;+\; \frac{x}{\sqrt{2\pi}} \;+\; O(x^3) \;\approx\; \frac{1}{2} \;+\; \frac{x}{\sqrt{2\pi}}. \tag{5.72}$$

[4] We note that the Taylor expansion (5.72) can be more accurately written as

$$N(x) \;=\; \frac{1}{2} \;+\; \frac{x}{\sqrt{2\pi}} \;-\; \frac{x^3}{6}\,\frac{1}{\sqrt{2\pi}} \;+\; O(x^5), \quad \text{as } x \to 0,$$

since $N'''(0) = -\frac{1}{\sqrt{2\pi}}$, $N^{(4)}(0) = 0$, and $N^{(5)}(0) = \frac{3}{\sqrt{2\pi}} \neq 0$. This level of precision is not needed here.

Thus,

$$N\left(\frac{\sigma\sqrt{T}}{2}\right) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\sigma\sqrt{T}}{2}, \tag{5.73}$$

and, by substituting (5.73) into (5.70), we find that

$$C = 2S\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1}{2}\right) \approx \sigma S\sqrt{\frac{T}{2\pi}},$$

which is the same as (5.66).

By solving (5.66) for σ, we obtain the following estimate for the implied volatility of an ATM call option:

$$\sigma_{imp} \approx \sqrt{2\pi}\frac{C}{S\sqrt{T}} \approx \frac{2.5\ C}{S\sqrt{T}}; \tag{5.74}$$

see sections 3.6.2 and 8.6 for more details on implied volatility.

Formula (5.66) is a good approximation for the Black–Scholes value of ATM options only if the Taylor approximation (5.72) is accurate. In section 5.5.3, we show that, for options with maturities less than one year and volatility slightly smaller than 50%, the approximation given by (5.66) is within one percent of the Black–Scholes option price; see Theorem 5.7 for the precise result.

The following approximations of the Black–Scholes formulas for at–the–money options if $r \neq 0$ and $q \neq 0$, i.e.,

$$C \approx \sigma S\sqrt{\frac{T}{2\pi}}\left(1 - \frac{(r+q)T}{2}\right) + \frac{(r-q)T}{2}S; \tag{5.75}$$

$$P \approx \sigma S\sqrt{\frac{T}{2\pi}}\left(1 - \frac{(r+q)T}{2}\right) - \frac{(r-q)T}{2}S, \tag{5.76}$$

can be obtained similarly. We only derive (5.75) here. For an ATM call option, the Black–Scholes formula (3.53) corresponding to $t = 0$ becomes

$$C = Se^{-qT}N\left(\frac{(r-q)\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}\right) - Se^{-rT}N\left(\frac{(r-q)\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}\right).$$

From the Taylor expansion (5.72) for $N(x)$, i.e., $N(x) \approx \frac{1}{2} + \frac{x}{\sqrt{2\pi}}$, we find that

$$C = Se^{-qT}\left(\frac{1}{2} + \frac{(r-q)\sqrt{T}}{\sigma\sqrt{2\pi}} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}}\right)$$

$$
\begin{aligned}
& - \; Se^{-rT} \left(\frac{1}{2} \; + \; \frac{(r-q)\sqrt{T}}{\sigma\sqrt{2\pi}} \; - \; \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} \right) \\
= \; & S \; \left(\frac{1}{2} \; + \; \frac{(r-q)\sqrt{T}}{\sigma\sqrt{2\pi}} \right) \; \left(e^{-qT} - e^{-rT} \right) \; + \; \sigma S \sqrt{\frac{T}{2\pi}} \; \frac{e^{-qT} + e^{-rT}}{2}.
\end{aligned}
$$

Using the quadratic Taylor approximations (5.20) for e^{-qT} and e^{-rT}, i.e., $e^{-qT} \approx 1 - qT + \frac{q^2T^2}{2}$ and $e^{-rT} \approx 1 - rT + \frac{r^2T^2}{2}$, we find that

$$
\begin{aligned}
e^{-qT} - e^{-rT} \; &= \; (r-q)T \; - \; \frac{(r^2-q^2)T^2}{2}; \\
\frac{e^{-qT} + e^{-rT}}{2} \; &= \; 1 - \frac{(r+q)T}{2} + \frac{(r^2+q^2)T^2}{4},
\end{aligned}
$$

and obtain that

$$
\begin{aligned}
C \; \approx \; & S\frac{(r-q)T}{2} + S\frac{(r-q)^2T\sqrt{T}}{\sigma\sqrt{2\pi}} - S\frac{(r^2-q^2)T^2}{4} \\
& - S\frac{(r-q)(r^2-q^2)T^2\sqrt{T}}{2\sigma\sqrt{2\pi}} + \sigma S\sqrt{\frac{T}{2\pi}} \left(1 - \frac{(r+q)T}{2} + \frac{(r^2+q^2)T^2}{4} \right).
\end{aligned}
$$

When ignoring the terms involving r^2 and q^2, which are, in general, much smaller than the other terms, the approximation formula (5.75) is obtained.

Solving for σ in (5.75), the following estimate for the implied volatility of ATM calls when $r \neq 0$ and $q \neq 0$ is found:

$$
\sigma_{imp} \; \approx \; \frac{\sqrt{2\pi}}{S\sqrt{T}} \; \frac{C \; - \; \frac{(r-q)T}{2} S}{1 - \frac{(r+q)T}{2}}. \tag{5.77}
$$

To obtain the ATM approximations of the Greeks of ATM plain vanilla European options, we use the closed formulas (3.66–3.75). We only present the formulas for the case $r = q = 0$. For the general case $r \neq 0$ and $q \neq 0$, the approximation formulas for the Greeks are obtained similarly.

If $K = S$ and $r = q = 0$, and for $t = 0$, the formulas (3.55) and (3.56) become

$$
d_1 \; = \; \frac{\sigma\sqrt{T}}{2} \quad \text{and} \quad d_2 \; = \; - \, d_1. \tag{5.78}
$$

Recall that

$$
\frac{1}{\sqrt{2\pi}} \; = \; 0.39894228 \; \approx \; 0.4. \tag{5.79}
$$

From (5.72), i.e., $N(x) \approx \frac{1}{2} + \frac{x}{\sqrt{2\pi}}$, we find that

$$N(d_1) \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} = \frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} \approx \frac{1}{2} + 0.2\sigma\sqrt{T}; \quad (5.80)$$

$$N(-d_1) \approx \frac{1}{2} - \frac{d_1}{\sqrt{2\pi}} = \frac{1}{2} - \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} \approx \frac{1}{2} - 0.2\sigma\sqrt{T}. \quad (5.81)$$

From (5.78) and the first order Taylor approximation (5.15) for the function e^x, we obtain that

$$e^{-\frac{d_1^2}{2}} \approx 1 - \frac{d_1^2}{2} = 1 - \frac{\sigma^2 T}{8}. \quad (5.82)$$

Using (5.79–5.82), we conclude from (3.66–3.75) for $t = 0$, $K = S$ and $r = q = 0$ that

$$\begin{aligned}
\Delta(C) &= N(d_1) \approx \frac{1}{2} + 0.2\sigma\sqrt{T};\\
\Delta(P) &= -N(-d_1) \approx -\frac{1}{2} + 0.2\sigma\sqrt{T};\\
\Gamma(C) = \Gamma(P) &= \frac{1}{S\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \approx \frac{0.4}{S\sigma\sqrt{T}}\left(1 - \frac{\sigma^2 T}{8}\right);\\
\mathrm{vega}(C) = \mathrm{vega}(P) &= S\sqrt{T}\,\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \approx 0.4S\sqrt{T}\left(1 - \frac{\sigma^2 T}{8}\right);\\
\Theta(C) = \Theta(P) &= -\frac{\sigma S}{2\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \approx -\frac{0.2\sigma S}{\sqrt{T}}\left(1 - \frac{\sigma^2 T}{8}\right);\\
\rho(C) &= ST\,N(-d_1) \approx ST\left(\frac{1}{2} - 0.2\sigma\sqrt{T}\right);\\
\rho(P) &= -ST\,N(d_1) \approx -ST\left(\frac{1}{2} + 0.2\sigma\sqrt{T}\right).
\end{aligned}$$

5.5.3 The precision of the ATM approximation of the Black–Scholes formula

A theoretical estimate for when the approximate value given by formula (5.66) for the price of the option of an ATM call on a non–dividend–paying asset is derived in Theorem 5.7. The following result will be needed in the proof of Theorem 5.7:

Lemma 5.1. *Let $N(x)$ be the cumulative density function of the standard normal variable. Then,*

$$N\left(\frac{\sigma\sqrt{T}}{2}\right) = \frac{1}{2} + \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\sigma\sqrt{T}}{2}} \left(\frac{\sigma\sqrt{T}}{2} - t\right) t\, e^{-\frac{t^2}{2}}\, dt. \quad (5.83)$$

Proof. For $n = 1$, formula (5.4) for the integral form of the Taylor approximation becomes

$$f(x) - f(a) - (x-a)f'(a) = \int_a^x (x-t)\, f''(t)\, dt, \quad (5.84)$$

since $P_1(x) = f(a) + (x-a)f'(a)$. By writing (5.84) for $f(x) = N(x)$ and $a = 0$, we obtain

$$N(x) - N(0) - xN'(0) = \int_0^x (x-t)\, N''(t)\, dt.$$

Recall that $N(0) = \frac{1}{2}$, $N'(0) = \frac{1}{\sqrt{2\pi}}$, and $N''(t) = -\frac{1}{\sqrt{2\pi}} t e^{-\frac{t^2}{2}}$. Then,

$$N(x) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} \int_0^x (x-t)\, t\, e^{-\frac{t^2}{2}}\, dt. \quad (5.85)$$

Formula (5.83) follows from (5.85), by substituting $x = \frac{\sigma\sqrt{T}}{2}$. □

Theorem 5.7. *Consider an at–the–money call option on a non–dividend paying asset, and assume zero interest rates. Let $C_{BS,r=q=0}$ be the Black–Scholes value of the call, and let $C_{approx,r=q=0}$ be the approximate value of the call option given by (5.66). The relative error given by the approximation formula (5.66) for ATM calls on non–dividend–paying assets is less than 1%, i.e.,*

$$\frac{|C_{approx,r=q=0} - C_{BS,r=q=0}|}{C_{BS,r=q=0}} \leq 0.01, \quad (5.86)$$

provided that the total variance $\sigma^2 T$ is bounded as follows:

$$\sigma^2 T \leq \frac{24}{101} \approx 0.2376.$$

Proof. Recall from (5.66) that

$$C_{approx,r=q=0} = \sigma S \frac{\sqrt{T}}{\sqrt{2\pi}}. \quad (5.87)$$

From (5.70), and using (5.83), we find that

$$\begin{aligned} C_{BS,r=q=0} &= 2S\left(N\left(\frac{\sigma\sqrt{T}}{2}\right)-\frac{1}{2}\right) \\ &= S\frac{\sigma\sqrt{T}}{\sqrt{2\pi}} - \frac{2S}{\sqrt{2\pi}}\int_0^{\frac{\sigma\sqrt{T}}{2}}\left(\frac{\sigma\sqrt{T}}{2}-t\right)t\ e^{-\frac{t^2}{2}}\ dt \\ &= C_{approx,r=q=0} - \frac{2S}{\sqrt{2\pi}}\int_0^{\frac{\sigma\sqrt{T}}{2}}\left(\frac{\sigma\sqrt{T}}{2}-t\right)t\ e^{-\frac{t^2}{2}}\ dt. \end{aligned} \quad (5.88)$$

From (5.88), we obtain that $C_{BS,r=q=0} < C_{approx,r=q=0}$. Then, condition (5.86) is equivalent to

$$\frac{|C_{approx,r=q=0} - C_{BS,r=q=0}|}{C_{BS,r=q=0}} = \frac{C_{approx,r=q=0} - C_{BS,r=q=0}}{C_{BS,r=q=0}} \le 0.01,$$

and therefore to

$$C_{BS,r=q=0} \ge \frac{1}{1.01}C_{approx,r=q=0}. \quad (5.89)$$

Using again (5.88) and (5.87), the inequality (5.89) can be written as

$$\frac{2S}{\sqrt{2\pi}}\int_0^{\frac{\sigma\sqrt{T}}{2}}\left(\frac{\sigma\sqrt{T}}{2}-t\right)t\ e^{-\frac{t^2}{2}}\ dt \le \frac{0.01}{1.01}C_{approx,r=q=0} = \frac{1}{101}\ \sigma\sqrt{T}\frac{S}{\sqrt{2\pi}}. \quad (5.90)$$

Since $e^{-\frac{t^2}{2}} < 1$, it is easy to see that

$$\int_0^{\frac{\sigma\sqrt{T}}{2}}\left(\frac{\sigma\sqrt{T}}{2}-t\right)t\ e^{-\frac{t^2}{2}}\ dt < \int_0^{\frac{\sigma\sqrt{T}}{2}}\left(\frac{\sigma\sqrt{T}}{2}-t\right)t\ dt = \frac{(\sigma\sqrt{T})^3}{48}.$$

Therefore, the inequality (5.90) is satisfied provided that

$$\frac{(\sigma\sqrt{T})^3}{48} \le \frac{1}{101}\ \frac{\sigma\sqrt{T}}{2}.$$

This is the same as $\sigma^2 T \le \frac{24}{101}$, which is what we wanted to prove. □

We conclude by investigating how good the approximations from section 5.5.1 are, if $r = 0$ and $q = 0$, and if $r \neq 0$ and $q \neq 0$, for two different ATM options.

Example: Estimate the relative errors given by the approximation formulas from section 5.5.1 for the value, the Greeks, and the implied volatility of