1.12 Exercises

1. Use the integration by parts to compute $\int \ln(x)\ dx$.

2. Compute $\int \frac{1}{x\ln(x)}\ dx$ by using the substitution $u = \ln(x)$.

3. Show that $(\tan x)' = 1/(\cos x)^2$ and

$$\int \frac{1}{1+x^2}\ dx \ = \ \arctan(x) \ + \ C.$$

   Note: The antiderivative of a rational function is often computed using the substitution $x = \tan\left(\frac{z}{2}\right)$.

4. Use l'Hôpital's rule to show that the following two Taylor approximations hold when x is close to 0:

$$\sqrt{1+x} \ \approx \ 1 \ + \ \frac{x}{2};$$
$$e^x \ \approx \ 1 \ + \ x \ + \ \frac{x^2}{2}.$$

   In other words, show that the following limits exist and are constant:

$$\lim_{x\to 0} \frac{\sqrt{1+x} \ - \ \left(1+\frac{x}{2}\right)}{x^2} \quad \text{and} \quad \lim_{x\to 0} \frac{e^x \ - \ \left(1+x+\frac{x^2}{2}\right)}{x^3}.$$

5. Use the definition (1.32) of e, i.e., $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, to show that

$$\frac{1}{e} \ = \ \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x.$$

   Hint: Use the fact that

$$\frac{1}{1+\frac{1}{x}} \ = \ \frac{x}{x+1} \ = \ 1 - \frac{1}{x+1}.$$

6. Let K, T, σ and r be positive constants, and define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) \ = \ \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{y^2}{2}}\ dy,$$

where $b(x) = \left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T\right) / \left(\sigma\sqrt{T}\right)$. Compute $g'(x)$.

Note: This function is related to the Delta of a plain vanilla Call option; see Section 3.5 for more details.

7. Let $f(x)$ be a continuous function. Show that

$$\lim_{h \to 0} \frac{1}{2h} \int_{a-h}^{a+h} f(x)\, dx = f(a), \quad \forall \; a \in \mathbb{R}.$$

Note: Let $F(x) = \int f(x)\, dx$. The central finite difference approximation (6.7) of $F'(a)$ is

$$F'(a) = \frac{F(a+h) - F(a-h)}{2h} + O\left(h^2\right), \qquad (1.54)$$

as $h \to 0$ (if $F^{(3)}(x) = f''(x)$ is continuous). Since $F'(a) = f(a)$, formula (1.54) can be written as

$$f(a) = \frac{1}{2h} \int_{a-h}^{a+h} f(x)\, dx + O(h^2).$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(y) = \sum_{i=1}^{n} c_i e^{-yt_i}$, where c_i and t_i, $i = 1 : n$, are positive constants. Compute $f'(y)$ and $f''(y)$.

Note: The function $f(y)$ represents the price of a bond with cash flows c_i paid at time t_i as a function of the yield y of the bond. When scaled appropriately, the derivative of $f(y)$ with respect to y give the duration and convexity of the bond; see Section 2.7 for more details.

9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = 2x_1^2 - x_1 x_2 + 3x_2 x_3 - x_3^2$, where $x = (x_1, x_2, x_3)$.

(i) Compute the gradient and Hessian of the function $f(x)$ at the point $a = (1, -1, 0)$, i.e., compute $Df(1, -1, 0)$ and $D^2 f(1, -1, 0)$.

(ii) Show that

$$f(x) = f(a) + Df(a)\,(x - a) + \frac{1}{2}\,(x - a)^t\, D^2 f(a)\,(x - a). \qquad (1.55)$$

Here, x, a, and $x - a$ are 3×1 column vectors, i.e.,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad a = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad x - a = \begin{pmatrix} x_1 - 1 \\ x_2 + 1 \\ x_3 \end{pmatrix}.$$

Note: Formula (1.55) is the quadratic Taylor approximation of $f(x)$ around the point a; cf. (5.32). Since $f(x)$ is a second order polynomial, the quadratic Taylor approximation of $f(x)$ is exact.

10. Let

$$u(x,t) \;=\; \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad \text{for} \;\; t > 0, \; x \in \mathbb{R}.$$

Compute $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, and show that

$$\frac{\partial u}{\partial t} \;=\; \frac{\partial^2 u}{\partial x^2}.$$

Note: This exercise shows that the function $u(x,t)$ is a solution of the heat equation. In fact, $u(x,t)$ is the fundamental solution of the heat equation, and is used in the PDE derivation of the Black–Scholes formula for pricing European plain vanilla options.

Also, note that $u(x,t)$ is the same as the density function of a normal variable with mean 0 and variance $2t$; cf. (3.48) for $\mu = 0$ and $\sigma^2 = 2t$.

11. Consider a portfolio with the following positions:

• long one call option with strike $K_1 = 30$;

• short two call options with strike $K_2 = 35$;

• long one call option with strike $K_3 = 40$.

All options are on the same underlying asset and have maturity T. Draw the payoff diagram at maturity of the portfolio, i.e., plot the value of the portfolio $V(T)$ at maturity as a function of $S(T)$, the price of the underlying asset at time T.

Note: This is a butterfly spread. A trader takes a long position in a butterfly spread if the price of the underlying asset at maturity is expected to be in the $K_1 \leq S(T) \leq K_3$ range.

12. Draw the payoff diagram at maturity of a bull spread with a long position in a call with strike 30 and short a call with strike 35, and of a bear spread with long a put of strike 20 and short a put of strike 15.

13. Which of the following two portfolios would you rather hold:

• Portfolio 1: Long one call option with strike $K = X - 5$ and long one call option with strike $K = X + 5$;

- Portfolio 2: Long two call options with strike $K = X$?

(All options are on the same asset and have the same maturity.)

14. Call options with strikes 100, 120, and 130 on the same underlying asset and with the same maturity are trading for 8, 5, and 3, respectively (there is no bid–ask spread). Is there an arbitrage opportunity present? If yes, how can you make a riskless profit?

15. A stock with spot price 40 pays dividends continuously at a rate of 3%. The four months at-the-money put and call options on this asset are trading at \$2 and \$4, respectively. The risk-free rate is constant and equal to 5% for all times. Show that the Put-Call parity is not satisfied and explain how would you take advantage of this arbitrage opportunity.

16. The bid and ask prices for a six months European call option with strike 40 on a non–dividend–paying stock with spot price 42 are \$5 and \$5.5, respectively. The bid and ask prices for a six months European put option with strike 40 on the same underlying asset are \$2.75 and \$3.25, respectively. Assume that the risk free rate is equal to 0. Is there an arbitrage opportunity present?

17. You expect that an asset with spot price \$35 will trade in the \$40–\$45 range in one year. One year at–the–money calls on the asset can be bought for \$4. To act on the expected stock price appreciation, you decide to either buy the asset, or to buy ATM calls. Which strategy is better, depending on where the asset price will be in a year?

18. The risk free rate is 8% compounded continuously and the dividend yield of a stock index is 3%. The index is at 12,000 and the futures price of a contract deliverable in three months is 12,100. Is there an arbitrage opportunity, and how do you take advantage of it?