8.9 Exercises

1. Find the maximum and minimum of the function $f(x_1, x_2, x_3) = 4x_2 - 2x_3$ subject to the constraints $2x_1 - x_2 - x_3 = 0$ and $x_1^2 + x_2^2 = 13$.

2. Assume that you can trade four assets (and that it is also possible to short the assets). The expected values, standard deviations, and correlations of the rates of return of the assets are:

$$\begin{aligned}
\mu_1 &= 0.08; & \sigma_1 &= 0.25; & \rho_{1,2} &= -0.25; \\
\mu_2 &= 0.12; & \sigma_2 &= 0.25; & \rho_{2,3} &= -0.25; \\
\mu_3 &= 0.16; & \sigma_3 &= 0.30; & \rho_{1,3} &= 0.25; \\
\mu_4 &= 0.05; & \sigma_4 &= 0.20; & \rho_{i,4} &= 0, \quad \forall\, i = 1:3.
\end{aligned}$$

   (i) Find the asset allocation for a minimal variance portfolio with 12% expected rate of return;

   (ii) Find the asset allocation for a maximum expected return portfolio with standard deviation of the rate of return equal to 24%.

3. Use Newton's method to find the yield of a five year semiannual coupon bond with 3.375% coupon rate and price $100\,\frac{1}{32}$. What are the duration and convexity of the bond?

4. Recall that finding the implied volatility from the given price of a call option is equivalent to solving the nonlinear problem $f(x) = 0$, where

$$f(x) \;=\; Se^{-qT}N(d_1(x)) - Ke^{-rT}N(d_2(x)) \;-\; C$$

   and $d_1(x) \;=\; \frac{\ln\left(\frac{S}{K}\right) + \left(r-q+\frac{x^2}{2}\right)T}{x\sqrt{T}}$, $d_2(x) \;=\; \frac{\ln\left(\frac{S}{K}\right) + \left(r-q-\frac{x^2}{2}\right)T}{x\sqrt{T}}$.

   (i) Show that $\lim_{x\to\infty} d_1(x) = \infty$ and $\lim_{x\to\infty} d_2(x) = -\infty$, and conclude that

$$\lim_{x\to\infty} f(x) \;=\; Se^{-qT} \;-\; C.$$

   (ii) Show that

$$\lim_{x\searrow 0} d_1(x) \;=\; \lim_{x\searrow 0} d_2(x) \;=\; \begin{cases} -\infty, & \text{if } Se^{(r-q)T} \;<\; K; \\ 0, & \text{if } Se^{(r-q)T} \;=\; K; \\ \infty, & \text{if } Se^{(r-q)T} \;>\; K. \end{cases}$$

   (Recall that $F = Se^{(r-q)T}$ is the forward price.)

Conclude that

$$\lim_{x \searrow 0} f(x) = \begin{cases} -C, & \text{if } Se^{(r-q)T} \leq K; \\ Se^{-qT} - Ke^{-rT} - C, & \text{if } Se^{(r-q)T} > K \end{cases}$$

(iii) Show that $f(x)$ is a strictly increasing function and

$$\begin{aligned} -C &< f(x) < Se^{-qT} - C, \quad \text{if } Se^{(r-q)T} \leq K; \\ Se^{-qT} - Ke^{-rT} - C &< f(x) < Se^{-qT} - C, \quad \text{if } Se^{(r-q)T} > K. \end{aligned}$$

(iv) For what range of call option values does the problem $f(x) = 0$ have a positive solution? Compare your result to the range given in (3.92).

5. A three months at–the–money call on an underlying asset with spot price 30 paying dividends continuously at a 2% rate is worth \$2.5. Assume that the risk free interest rate is constant at 6%.

   (i) Compute the implied volatility with six decimal digits accuracy, using the bisection method on the interval $[0.0001, 1]$, the secant method with initial guess 0.5, and Newton's method with initial guess 0.5.

   (ii) Let σ_{imp} be the implied volatility previously computed using Newton's method. Use formula (5.77) to compute an approximate value $\sigma_{imp,approx}$ for the implied volatility, and compute the relative error

$$\frac{|\sigma_{imp,approx} - \sigma_{imp}|}{\sigma_{imp}}.$$

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(x) = \begin{pmatrix} x_1^3 + 2x_1x_2 + x_3^2 - x_2x_3 + 9 \\ 2x_1^2 + 2x_1x_2^2 + x_2^3x_3^2 - x_2^2x_3 - 2 \\ x_1x_2x_3 + x_1^3 - x_3^2 - x_1x_2^2 - 4 \end{pmatrix}.$$

   The approximate gradient $\Delta_c F(x) = (\Delta_{c,j} F_i(x))_{i,j=1:n}$ of $F(x)$ is computed using central difference approximations, i.e.,

$$\Delta_{c,j} F_i(x) = \frac{F_i(x + he_j) - F_i(x - he_j)}{2h}, \quad j = 1 : n,$$

   where e_j is a vector with all entries equal to 0 with the exception of the j-th entry, which is equal to 1.

(i) Solve $F(x) = 0$ using the approximate Newton's algorithm from Table 8.4, by substituting $\Delta_c F(x_{old})$ for $\Delta F(x_{old})$. Use $h = 10^{-6}$, tol_consec $= 10^{-6}$, and tol_approx $= 10^{-9}$, and two different initial guesses:

$$x_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

(ii) Compare these results to those obtained in section 8.3.2 corresponding to the approximate Newton's method with forward finite difference approximations for $\Delta F(x)$.

7. (i) Use bootstrapping to obtain a zero rate curve from the following prices of Treasury instruments with semiannual coupon payments:

	Coupon Rate	Price
3 – Month T-bill	0	98.7
6 – Month T-bill	0	97.5
2 – Year T-bond	4.875	$100\frac{5}{32}$
3 – Year T-bond	4.875	$100\frac{5}{32}$
5 – Year T-bond	4.625	$99\frac{22}{32}$
10 – Year T-bond	4.875	$101\frac{4}{32}$

Assume that interest is continuously compounded.

(ii) How would the zero rate curves obtained by bootstrapping from the bond prices above, one corresponding to semi-annually compounded interest, and the other one corresponding to continuously computed interest, compare? In other words, will one of the two curves be higher or lower than the other one, and why?

8. Use bootstrapping to obtain a zero rate curve given the prices of the following semiannual coupon bonds:

Maturity	Coupon Rate	Price
6 months	0	97.5
1 year	5	100
20 months	6	103
40 months	5	102
5 years	4	103

Assume that the overnight rate is 5% and that the zero rate curve is linear on the following time intervals:

$$[0, 0.5]; \quad [0.5, 1]; \quad \left[1, \frac{5}{3}\right]; \quad \left[\frac{5}{3}, \frac{10}{3}\right]; \quad \left[\frac{10}{3}, 5\right].$$

Hint: Obtain the zero rate curve $r(0,t)$ for $0 \le t \le 1$ from the prices of the first two bonds, and using the fact that $r(0,0) = 0.05$. The third bond pays coupons in 2, 8, 14, and 20 months, when it also pays the face value of the bond. From the pricing formula (8.72), it follows that

$$103 \;=\; 3\,e^{-\frac{1}{6}r(0,\frac{1}{6})} \;+\; 3\,e^{-\frac{2}{3}r(0,\frac{2}{3})} \;+\; 3\,e^{-\frac{7}{6}r(0,\frac{7}{6})} \;+\; 103\,e^{-\frac{5}{3}r(0,\frac{5}{3})}. \quad (8.79)$$

The zero rates $r(0,\frac{1}{6})$ and $r(0,\frac{2}{3})$ are already known. Since the zero rate curve is assumed to be linear for $t \in \left[1, \frac{5}{3}\right]$, we find that

$$r(0,t) \;=\; \frac{3}{2}\left(\frac{5}{3} - t\right)\, r(0,1) \;+\; \frac{3}{2}\,(t-1)\; r\left(0, \frac{5}{3}\right).$$

Therefore,

$$r\left(0, \frac{7}{6}\right) \;=\; \frac{r(0,1)}{3} \;+\; \frac{r\left(0, \frac{5}{3}\right)}{4}.$$

Set $x = r(0,\frac{5}{3})$ and use Newton's method to solve for x in (8.79).