While definitions (3.61–3.65) are valid for the Greeks of any option on one underlying asset, including, e.g., American and exotic options, closed formulas for the Greeks can only be obtained if closed formulas for pricing the options exist. This happens very rarely for options that are not plain vanilla European, a notable exception being European barrier options; see section 7.8.

3.6.1 Explaining the magic of Greeks computations

It is interesting to note that the formulas (3.66–3.75) for the Greeks are simpler than expected. For example, the Delta of a call option is defined as

$$\Delta(C) = \frac{\partial C}{\partial S}.$$

Differentiating the Black–Scholes formula (3.53) with respect to S, we obtain

$$\begin{aligned}\Delta(C) &= e^{-q(T-t)}N(d_1)\\ &\quad + Se^{-q(T-t)}\frac{\partial}{\partial S}(N(d_1)) - Ke^{-r(T-t)}\frac{\partial}{\partial S}(N(d_2)), \qquad (3.77)\end{aligned}$$

since both d_1 and d_2 are functions of S; cf. (3.55) and (3.56).

However, we know from (3.66) that

$$\Delta(C) = e^{-q(T-t)}N(d_1). \qquad (3.78)$$

To understand how (3.77) reduces to (3.78), we apply chain rule and obtain that

$$\frac{\partial}{\partial S}(N(d_1)) = N'(d_1)\frac{\partial d_1}{\partial S}; \qquad (3.79)$$

$$\frac{\partial}{\partial S}(N(d_2)) = N'(d_2)\frac{\partial d_2}{\partial S}. \qquad (3.80)$$

Then, using (3.79) and (3.80), we can write (3.77) as

$$\begin{aligned}\Delta(C) &= e^{-q(T-t)}N(d_1)\\ &\quad + Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}. \qquad (3.81)\end{aligned}$$

Note that formulas (3.55) and (3.56) for d_1 and d_2 can be written as

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}; \qquad (3.82)$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}. \qquad (3.83)$$

The following result explains why (3.81) reduces to (3.78):

Lemma 3.15. *Let d_1 and d_2 be given by (3.82) and (3.83). Then*

$$Se^{-q(T-t)}\ N'(d_1) \ = \ Ke^{-r(T-t)}\ N'(d_2). \tag{3.84}$$

Proof. Recall that $N(z)$ is the cumulative distribution of the standard normal variable, i.e.,

$$N(z) \ = \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\ dx.$$

From Lemma 2.3, we find that $N'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Then,

$$N'(d_1) \ = \ \frac{1}{\sqrt{2\pi}}\ e^{\frac{-d_1^2}{2}}; \tag{3.85}$$

$$N'(d_2) \ = \ \frac{1}{\sqrt{2\pi}}\ e^{\frac{-d_2^2}{2}}. \tag{3.86}$$

Therefore, in order to prove (3.84), it is enough to show that the following formula holds true:

$$Se^{-q(T-t)}\ e^{-\frac{d_1^2}{2}} \ = \ Ke^{-r(T-t)}\ e^{-\frac{d_2^2}{2}}. \tag{3.87}$$

Recall the notation $\exp(x) = e^x$. Since $\exp(\ln(x)) = x$, we find that

$$\begin{aligned} Se^{-q(T-t)} \ &= \ Ke^{-r(T-t)}\ \frac{S}{K} e^{(r-q)(T-t)} \\ &= \ Ke^{-r(T-t)}\ \exp\left(\ln\left(\frac{S}{K}\right) + (r-q)(T-t)\right). \end{aligned}$$

From (3.82), it is easy to see that

$$\ln\left(\frac{S}{K}\right) + (r-q)(T-t) \ = \ d_1\sigma\sqrt{T-t} \ - \ \frac{\sigma^2(T-t)}{2},$$

and therefore

$$Se^{-q(T-t)} \ = \ Ke^{-r(T-t)}\ \exp\left(d_1\sigma\sqrt{T-t} \ - \ \frac{\sigma^2(T-t)}{2}\right).$$

Using the fact that $d_2 = d_1 - \sigma\sqrt{T-t}$, we obtain (3.87) as follows:

$$\begin{aligned} Se^{-q(T-t)}\ e^{-\frac{d_1^2}{2}} \ &= \ Ke^{-r(T-t)}\ \exp\left(-\frac{d_1^2}{2} \ + \ d_1\sigma\sqrt{T-t} \ - \ \frac{\sigma^2(T-t)}{2}\right) \\ &= \ Ke^{-r(T-t)}\ \exp\left(-\frac{(d_1 - \sigma\sqrt{T-t})^2}{2}\right) \\ &= \ Ke^{-r(T-t)}\ e^{-\frac{d_2^2}{2}}. \end{aligned}$$

□

We return our attention to proving formula (3.78) for $\Delta(C)$.

From (3.82) and (3.83), we find that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}}. \tag{3.88}$$

Using (3.88) and Lemma 3.15, we conclude that formula (3.81) becomes

$$\begin{aligned}\Delta(C) &= e^{-q(T-t)}N(d_1) + Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} \\ &= e^{-q(T-t)}N(d_1) + Se^{-q(T-t)}\,N'(d_1)\left(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right) \\ &= e^{-q(T-t)}N(d_1).\end{aligned}$$

Formula (3.78) is therefore proven.

The simplified formulas (3.70), (3.72), and (3.74) for the vega, Θ, and ρ of a European call option[6] are obtained similarly using Lemma 3.15.

The formula for vega(C):

We differentiate the Black–Scholes formula (3.53) with respect to σ. Following the same steps as in the computation for the Delta of the call option, i.e., using chain rule and Lemma 2.3, we obtain that

$$\text{vega}(C) = \frac{\partial C}{\partial \sigma} = Se^{-q(T-t)}\,N'(d_1)\,\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}\,N'(d_2)\,\frac{\partial d_2}{\partial \sigma}.$$

Using the result of Lemma 3.15, we conclude that

$$\text{vega}(C) = Se^{-q(T-t)}\,N'(d_1)\,\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right).$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$, we find that $d_1 - d_2 = \sigma\sqrt{T-t}$ and thus

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \sqrt{T-t}.$$

Then, using the fact that $N'(d_1) = \frac{1}{\sqrt{2\pi}}\,e^{\frac{-d_1^2}{2}}$, see (3.85), we conclude that

$$\text{vega}(C) = \frac{1}{\sqrt{2\pi}}\,Se^{-q(T-t)}\,e^{-\frac{d_1^2}{2}}\,\sqrt{T-t},$$

[6]Note that the formulas (3.71), (3.73), and (3.75) for the vega, Θ, and ρ of a European put option can be obtained from (3.70), (3.72), and (3.74) by using the Put–Call parity.

which is the same as formula (3.70).

The formula for $\Theta(C)$:
We differentiate the Black–Scholes formula (3.53) with respect to t. Using chain rule and Lemma 2.3 we obtain that

$$\begin{aligned}\Theta(C) &= Se^{-q(T-t)}\, N'(d_1)\, \frac{\partial d_1}{\partial t} + qSe^{-q(T-t)}N(d_1)\\ &\quad - Ke^{-r(T-t)}\, N'(d_2)\, \frac{\partial d_2}{\partial t} - rKe^{-r(T-t)}N(d_2).\end{aligned}$$

Using the result of Lemma 3.15, we conclude that

$$\begin{aligned}\Theta(C) &= Se^{-q(T-t)}\, N'(d_1) \left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)\\ &\quad + qSe^{-q(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2).\end{aligned}$$

Since $d_1 - d_2 = \sigma\sqrt{T-t}$, we find that

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}.$$

Since $N'(d_1) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{-d_1^2}{2}}$, cf. (3.85), we conclude that

$$\begin{aligned}\Theta(C) &= -\frac{1}{\sqrt{2\pi}}\, Se^{-q(T-t)}\, e^{-\frac{d_1^2}{2}}\, \frac{\sigma}{2\sqrt{T-t}}\\ &\quad + qSe^{-q(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2),\end{aligned}$$

which is the same as formula (3.72).

The formula for $\rho(C)$:
We differentiate the Black–Scholes formula (3.53) with respect to r. Using Chain Rule and Lemma 2.3 we obtain that

$$\rho(C) = Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r} + K(T-t)e^{-r(T-t)}N(d_2).$$

Using Lemma 3.15, we find that

$$\rho(C) = Se^{-q(T-t)}\, N'(d_1) \left(\frac{\partial d_1}{\partial r} - \frac{\partial d_2}{\partial r}\right) + K(T-t)e^{-r(T-t)}N(d_2).$$

Since $d_1 - d_2 = \sigma\sqrt{T-t}$, we find that

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r},$$

and therefore,

$$\rho(C) = K(T-t)e^{-r(T-t)}N(d_2),$$

which is the same as formula (3.74).