Quadratic Approximations:

$$e^x \approx 1 + x + \frac{x^2}{2}; \quad (5.20)$$

$$\ln(1+x) \approx x - \frac{x^2}{2}; \quad (5.21)$$

$$\ln(1-x) \approx -x - \frac{x^2}{2}; \quad (5.22)$$

$$\frac{1}{1+x} \approx 1 - x + x^2; \quad (5.23)$$

$$\frac{1}{1-x} \approx 1 + x + x^2. \quad (5.24)$$

Note that these approximations are accurate only for small values of x.

5.2 Taylor's formula for multivariable functions

Scalar Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables denoted by $x_1, x_2, \ldots, x_n$. Let $x = (x_1, x_2, \ldots, x_n)$. We present first order and second order Taylor expansions of the function $f(x)$, without providing convergence results.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. The linear Taylor expansion of the function $f(x)$ around the point a,

$$f(x) \approx f(a) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a), \quad (5.25)$$

is a second order approximation, in the sense that

$$f(x) = f(a) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_n) + O\left(||x-a||^2\right), \quad (5.26)$$

as $x \to a$, if all the partial derivatives of second order of $f(x)$, i.e., $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$, $1 \le i, j \le n$, are continuous. Here, $O\left(||x-a||^2\right) = \sum_{i=1}^{n} O\left(|x_i - a_i|^2\right)$.

The quadratic Taylor expansion of the function $f(x)$ around the point a,

$$\begin{aligned} f(x) \approx\ & f(a) + \sum_{i=1}^{n} (x_i - a_i) \frac{\partial f}{\partial x_i}(a) \\ & + \sum_{1 \le i,j \le n} \frac{(x_i - a_i)(x_j - a_j)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(a), \end{aligned} \quad (5.27)$$

is a third order approximation, in the sense that

$$
\begin{aligned}
f(x) &= f(a) + \sum_{i=1}^{n}(x_i - a_i)\,\frac{\partial f}{\partial x_i}(a) \\
&\quad + \sum_{1\le i,j\le n} \frac{(x_i - a_i)(x_j - a_j)}{2}\,\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \\
&\quad + \sum_{i=1}^{n} O\left(|x_i - a_i|^3\right), \qquad (5.28)
\end{aligned}
$$

as $x \to a$, if all the third order partial derivatives of $f(x)$ are continuous.

The linear and quadratic Taylor expansions (5.25–5.28) can be written using matrix notation as follows: Recall from (1.36) that the gradient $Df(x)$ of the function $f(x)$ is a row vector of size n, i.e.,

$$
Df(x) = \left(\frac{\partial f}{\partial x_1}(x) \;\; \frac{\partial f}{\partial x_2}(x) \;\; \ldots \;\; \frac{\partial f}{\partial x_n}(x) \right),
$$

and, from (1.37), that the Hessian $D^2 f(x)$ is an $n \times n$ matrix:

$$
D^2 f(x) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\
\frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) & \ldots & \frac{\partial^2 f}{\partial x_n^2}(x)
\end{pmatrix}.
$$

Let

$$
x - a = \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix}. \qquad (5.29)
$$

The linear Taylor expansions (5.25) and (5.26) of $f(x)$ around the point a can be written in terms of the gradient $Df(a)$ as

$$
f(x) \approx f(a) + Df(a)\,(x-a); \qquad (5.30)
$$

$$
f(x) = f(a) + Df(a)\,(x-a) + O\left(||x-a||^2\right), \qquad (5.31)
$$

as $x \to a$.

The quadratic Taylor expansions (5.27) and (5.28) of $f(x)$ around the point a can be written in terms of $Df(a)$ and $D^2 f(a)$ as

$$
f(x) \approx f(a) + Df(a)\,(x-a) + \frac{1}{2}\,(x-a)^t\,D^2 f(a)\,(x-a); \qquad (5.32)
$$

$$\begin{aligned} f(x) &= f(a) + Df(a)\ (x-a) + \frac{1}{2}\ (x-a)^t\ D^2 f(a)\ (x-a) \\ &\quad + \sum_{i=1}^{n} O\left(|x_i - a_i|^3\right), \end{aligned} \tag{5.33}$$

as $x \to a$. Note that $(x-a)^t = (x_1 - a_1 \quad x_2 - a_2 \quad \ldots \quad x_n - a_n)$ is a row vector, the transpose of the column vector $x - a$ from (5.29).

Vector Valued Functions

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function, given by

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix},$$

where $x = (x_1, x_2, \ldots, x_n)$. Recall from (1.38) that the gradient $DF(x)$ of $F(x)$ is a matrix operator of size $m \times n$ given by

$$DF(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \ldots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}.$$

The linear Taylor expansion of the function $F(x)$ around the point $a \in \mathbb{R}^n$,

$$F(x) \approx F(a) + DF(a)\ (x-a) \tag{5.34}$$

is a second order approximation, i.e.,

$$F(x) = F(a) + DF(a)\ (x-a) + O\left(||x-a||^2\right), \tag{5.35}$$

as $x \to a$, if all the partial order derivatives of order two of the functions $f_k(x)$, $k = 1 : m$, are continuous. As before, $O\left(||x-a||^2\right) = \sum_{i=1}^n O\left(|x_i - a_i|^2\right)$, and $x - a$ is the column vector given by (5.29).

We note that formula (5.34) can be written explicitely as

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} \approx \begin{pmatrix} f_1(a) + \sum_{i=1}^n \frac{\partial f_1}{\partial x_i}(a)\ (x_i - a_i) \\ f_2(a) + \sum_{i=1}^n \frac{\partial f_2}{\partial x_i}(a)\ (x_i - a_i) \\ \vdots \\ f_m(a) + \sum_{i=1}^n \frac{\partial f_m}{\partial x_i}(a)\ (x_i - a_i) \end{pmatrix}.$$

Thus, the linear Taylor approximation formula (5.34) for the vector valued function $F(x)$ is obtained by combining the linear Taylor approximation formula (5.25) for each function $f_k(x)$, $k = 1 : m$.

5.2.1 Taylor's formula for functions of two variables

For clarification purposes, we include the formulas for Taylor expansions of both scalar and vector valued functions of two variables in this section.

Scalar Functions

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. The linear Taylor expansion of $f(x,y)$ around the point $(a,b) \in \mathbb{R}^2$

$$f(x,y) \approx f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \tag{5.36}$$

is a second order approximation, i.e.,

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + O\left(|x-a|^2\right) + O\left(|y-b|^2\right), \end{aligned} \tag{5.37}$$

as $(x,y) \to (a,b)$, if $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$ are continuous functions.

The quadratic Taylor expansion of $f(x,y)$ around the point $(a,b) \in \mathbb{R}^2$,

$$\begin{aligned} f(x,y) &\approx f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{(x-a)^2}{2}\frac{\partial^2 f}{\partial x^2}(a,b) + (x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) \\ &\quad + \frac{(y-b)^2}{2}\frac{\partial^2 f}{\partial y^2}(a,b), \end{aligned} \tag{5.38}$$

is a third order approximation, i.e.,

$$\begin{aligned} f(x,y) &= f(a,b) + (x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b) \\ &\quad + \frac{(x-a)^2}{2}\frac{\partial^2 f}{\partial x^2}(a,b) + (x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) \\ &\quad + \frac{(y-b)^2}{2}\frac{\partial^2 f}{\partial y^2}(a,b) + O\left(|x-a|^3\right) + O\left(|y-b|^3\right), \end{aligned} \tag{5.39}$$

as $(x,y) \to (a,b)$, if all third order derivatives of $f(x,y)$ are continuous.

The matrix forms of the linear Taylor expansions (5.36) and (5.37) are

$$f(x,y) \approx f(a,b) + Df(a,b)\begin{pmatrix} x-a \\ y-b \end{pmatrix};$$

$$f(x,y) = f(a,b) + Df(a,b)\begin{pmatrix} x-a \\ y-b \end{pmatrix} + O\left(|x-a|^2\right) + O\left(|y-b|^2\right),$$

as $(x,y) \to (a,b)$, where $Df(a,b) = \left(\frac{\partial f}{\partial x}(a,b) \;\; \frac{\partial f}{\partial y}(a,b) \right)$.
The matrix forms of the quadratic Taylor expansions (5.38) and (5.39) are

$$\begin{aligned} f(x,y) &\approx f(a,b) + Df(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ &\quad + \frac{1}{2} \begin{pmatrix} x-a \\ y-b \end{pmatrix}^t D^2 f(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix}; \qquad (5.40) \\ f(x,y) &= f(a,b) + Df(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ &\quad + \frac{1}{2} \begin{pmatrix} x-a \\ y-b \end{pmatrix}^t D^2 f(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ &\quad + O\left(|x-a|^3\right) + O\left(|y-b|^3\right), \qquad (5.41) \end{aligned}$$

as $(x,y) \to (a,b)$, where

$$D^2 f(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix}.$$

Vector Valued Functions

We conclude by writing linear and quadratic Taylor expansions for a function of two variables taking values in $\mathbb{R}^2$. Let

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{given by} \quad F(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}.$$

Recall from (1.41) that the gradient $DF(x,y)$ of $F(x,y)$ is

$$DF(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix}.$$

The linear Taylor expansion of $F(x,y)$ around the point $(a,b) \in \mathbb{R}^2$,

$$F(x,y) \approx F(a,b) + DF(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix},$$

is a second order approximation:

$$F(x,y) = F(a,b) + DF(a,b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} + O\left(|x-a|^2\right) + O\left(|y-b|^2\right),$$

as $(x,y) \to (a,b)$.