Since $C_1 \geq C_2$ and since $(\Delta y)^2 \geq 0$, we conclude from (6.76) and (6.77) that, regardless of whether the yield goes up or down,

$$\frac{\Delta B_1}{B_1} \geq \frac{\Delta B_2}{B_2}.$$

In other words, the bond with higher convexity provides a higher return, and is the better investment, all other things being considered equal.

6.7 Dollar duration and dollar convexity

Modified duration and convexity are not well suited for analyzing bond portfolios since they are non–additive, i.e., the modified duration and the convexity of a portfolio made of positions in different bonds are not equal to the sum of the modified durations or of the convexities, respectively, of the bond positions. Dollar duration and dollar convexity[12] are additive and can be used to measure the sensitivity of bond portfolios with respect to parallel changes in the zero rate curve.

We begin by defining dollar duration and dollar convexity for a single bond, and we then extend the definitions to bond portfolios.

Definition 6.1. *The dollar duration of a bond is defined as*

$$D_{\$} = -\frac{\partial B}{\partial y}, \tag{6.78}$$

and measures the sensitivity of the bond price with respect to small changes of the bond yield.

Recall that the modified duration of a bond is $D = -\frac{1}{B}\frac{\partial B}{\partial y}$; cf. (6.71). Then, from (6.78), it is easy to see that

$$D_{\$} = BD. \tag{6.79}$$

Definition 6.2. *The dollar convexity of a bond is defined as*

$$C_{\$} = \frac{\partial^2 B}{\partial y^2} \tag{6.80}$$

and measures the sensitivity of the dollar duration of a bond with respect to small changes of the bond yield.

[12] Dollar duration and dollar convexity are also called value duration and value convexity, which is appropriate for financial instruments that are not USD denominated.

Note that

$$C_\$ \;=\; -\frac{\partial D_\$}{\partial y};$$

cf. (6.78) and (6.80). Also, since the convexity of a bond is $C = \frac{1}{B}\frac{\partial^2 B}{\partial y^2}$, see (6.71), it follows from (6.80) that

$$C_\$ \;=\; BC. \tag{6.81}$$

It is important to note that the formula (6.73) can be written in terms of the dollar duration and of the dollar convexity of the bond as follows:

Lemma 6.4. *Let $D_\$$ and $C_\$$ be the dollar duration and the dollar convexity of a bond with yield y and value $B = B(y)$. Then,*

$$\Delta B \;\approx\; -D_\$\Delta y \;+\; \frac{C_\$}{2}(\Delta y)^2. \tag{6.82}$$

where $\Delta B = B(y + \Delta y) - B(y)$.

Proof. The result of (6.82) follows immediately from (6.73) and the definitions (6.78) and (6.80). □

We now define the dollar duration and dollar convexity of bond portfolios.

Consider a bond portfolio made of (either long or short) positions in p different bonds, and let B_j, $j = 1 : p$, be the (positive or negative) values of these bond positions. Denote by

$$V \;=\; \sum_{j=1}^{p} B_j$$

the value of the bond portfolio.

Definition 6.3. *The dollar duration $D_\$(V)$ of a bond portfolio is the sum of the dollar durations of the underlying bond positions, i.e.,*

$$D_\$(V) \;=\; \sum_{j=1}^{p} D_\$(B_j), \tag{6.83}$$

where $D_\$(B_j)$ is the dollar duration of the bond position B_j, $j = 1 : p$.

Definition 6.4. *The dollar convexity $C_\$(V)$ of a bond portfolio is the sum of the dollar convexities of the underlying bond positions, i.e.,*

$$C_\$(V) \;=\; \sum_{j=1}^{p} C_\$(B_j), \tag{6.84}$$

where $C_\$(B_j)$ is the dollar convexity of the bond position B_j, $j = 1 : p$.

An approximation formula similar to (6.82) can be derived for the change in the value of the bond portfolio for small, parallel changes in the zero rate curve, in terms of the dollar duration and dollar convexity of the portfolio.

Lemma 6.5. *Consider a bond portfolio with value V and denote by $D_{\$}(V)$ and $C_{\$}(V)$ the dollar duration and the dollar convexity of the portfolio. If the zero rate curve experiences a small parallel shift of size δr, the corresponding change ΔV in the value of the bond portfolio can be approximated as follows:*

$$\Delta V \;\approx\; -\,D_{\$}(V)\delta r \;+\; \frac{C_{\$}(V)}{2}(\delta r)^2. \tag{6.85}$$

Proof. Denote by B_j, $j = 1 : p$, the bond positions from the bond portfolio. The value of the bond portfolio is $V = \sum_{j=1}^{p} B_j$.

If δr is the size of the small, parallel change in the zero rate curve, the zero rate curve changed from $r(0,t)$ to $r_1(0,t) = r(0,t) + \delta r$. Let $\Delta V = V_1 - V$ be the corresponding change in the value of the bond portfolio. Then,

$$\Delta V \;=\; \sum_{j=1}^{p} \Delta B_j, \tag{6.86}$$

where ΔB_j is the change in the value of the bond position B_j, for $j = 1 : p$.

For a small parallel shift of size δr in the zero rate curve, it follows from Lemma 6.2 that the change Δy_j in the yield of bond B_j is approximately of size δr, i.e., $\Delta y_j \approx \delta r$, for all $j = 1 : p$. Then, we can use (6.82) to conclude that

$$\begin{aligned}\Delta B_j \;&\approx\; -D_{\$}(B_j)\Delta y_j \;+\; \frac{C_{\$}(B_j)}{2}(\Delta y_j)^2\\ &\approx\; -D_{\$}(B_j)\delta r \;+\; \frac{C_{\$}(B_j)}{2}(\delta r)^2,\end{aligned} \tag{6.87}$$

where $D_{\$}(B_j)$ and $C_{\$}(B_j)$ denote the dollar duration and the dollar convexity of the bond position B_j.

From (6.86) and (6.87), and using the definitions (6.83) and (6.84) of the dollar duration and of the dollar convexity of a bond portfolio, we conclude that

$$\begin{aligned}\Delta V \;&=\; \sum_{j=1}^{p} \Delta B_j\\ &\approx\; -\delta r \sum_{j=1}^{p} D_{\$}(B_j) \;+\; \frac{1}{2}(\delta r)^2 \sum_{j=1}^{p} C_{\$}(B_j)\\ &=\; -D_{\$}(V)\delta r \;+\; \frac{C_{\$}(V)}{2}(\delta r)^2.\end{aligned}$$

□

6.7.1 DV01

DV01 stands for "dollar value of one basis point" and is often used instead of dollar duration when quoting the risk associated with a bond position or with a bond portfolio.

Definition 6.5. *The DV01 of a bond measures the change in the value of the bond for a decrease of one basis point (i.e., of 0.01%, or, equivalently, of 0.0001) in the yield of the bond, and is equal to the dollar duration of the bond divided by* 10,000, *i.e.,*

$$DV01 \;=\; \frac{D_\$}{10{,}000}. \tag{6.88}$$

The DV01 of a bond is always positive, since a decrease in the bond yield results in an increase in the value of the bond.

To better understand the definition (6.88) of DV01, note that the change ΔB in the value of a bond corresponding to a change $\Delta y = -0.0001$ in the value of the bond yield can be approximated as follows:

$$\Delta B \;\approx\; -\,D_\$\Delta y + \frac{C_\$}{2}(\Delta y)^2 \;\approx\; -\,D_\$\Delta y \;=\; D_\$ \cdot 0.0001; \tag{6.89}$$

cf. (6.82) for $\Delta y = -0.0001$. Since $D_\$ = 10{,}000 \cdot \mathrm{DV01}$, cf. (6.88), we conclude from (6.89) that

$$\Delta B \;\approx\; \mathrm{DV01}, \quad \text{for} \quad \Delta y = -0.0001.$$

Definition 6.6. *The DV01 of a bond portfolio measures the change in the value of the portfolio for a downward parallel shift of the zero rate curve of one basis point, and is equal to the dollar duration of the bond portfolio divided by* 10,000, *i.e.,*

$$DV01(V) \;=\; \frac{D_\$(V)}{10{,}000}. \tag{6.90}$$

The DV01 of a bond portfolio may be positive or negative, unlike the DV01 of a bond, which is always positive.

Note that, if the zero rate curve has a downward parallel shift of one basis point (corresponding to $\delta r = -0.0001$), the change ΔV in the value of a bond portfolio can be approximated as follows:

$$\Delta V \;\approx\; -\,D_\$(V)\delta r \;+\; \frac{C_\$(V)}{2}(\delta r)^2 \;\approx\; -\,D_\$(V)\delta r \;=\; D_\$(V)\cdot 0.0001;$$

cf. (6.85) for $\delta r = -0.0001$. Since $D_\$(V) = 10{,}000 \cdot \mathrm{DV01}(V)$, cf. (6.90), we conclude that

$$\Delta V \;\approx\; \mathrm{DV01}(V), \quad \text{for} \quad \delta r = -0.0001.$$

6.7.2 Bond portfolio immunization

To reduce the sensitivity of a bond portfolio with respect to small changes in the zero rate curve, it is desirable to have a portfolio with small dollar duration and dollar convexity. The process of obtaining a portfolio with zero dollar duration and dollar convexity is called portfolio immunization, and can be done by taking positions in other bonds available on the market[13].

Let V be the value of a portfolio with dollar duration $D_\$(V)$ and dollar convexity $C_\$(V)$. Take positions of sizes B_1 and B_2, respectively, in two bonds with duration and convexity D_i and C_i, for $i = 1 : 2$. The value of the new portfolio is

$$\Pi \;=\; V \;+\; B_1 \;+\; B_2.$$

Note that $D_\$(B_1) = B_1 D_1$; $D_\$(B_2) = B_2 D_2$; $C_\$(B_1) = B_1 C_1$; $C_\$(B_2) = B_2 C_2$. Choose B_1 and B_2 such that the dollar duration and dollar convexity of the portfolio Π are equal to 0, i.e., such that

$$\begin{aligned} D_\$(\Pi) &= D_\$(V) + D_\$(B_1) + D_\$(B_2) \;=\; D_\$(V) + B_1 D_1 + B_2 D_2 \;=\; 0; \\ C_\$(\Pi) &= C_\$(V) + C_\$(B_1) + C_\$(B_2) \;=\; C_\$(V) + B_1 C_1 + B_2 C_2 \;=\; 0. \end{aligned}$$

The resulting linear system for B_1 and B_2, i.e.,

$$\left\{ \begin{array}{rcl} B_1 D_1 + B_2 D_2 &=& -D_\$(V); \\ B_1 C_1 + B_2 C_2 &=& -C_\$(V), \end{array} \right.$$

has a unique solution if and only if

$$\frac{D_1}{C_1} \;\neq\; \frac{D_2}{C_2}.$$

In other words, it is possible to immunize a bond portfolio using any two bonds with different duration–to–convexity ratios.

Example: Invest \$1 million in a bond with duration 3.2 and convexity 16, and invest \$2.5 million in a bond with duration 4 and convexity 24.

(i) What are the dollar duration and the dollar convexity of your portfolio?

(ii) If the zero rate curve goes up by ten basis points, estimate the new value of the portfolio.

(iii) You can buy or sell two other bonds, one with duration 1.6 and convexity 12 and another one with duration 3.2 and convexity 20. What positions would you take in these bonds to immunize your portfolio, i.e., to obtain a portfolio with zero dollar duration and dollar convexity?

Answer: (i) The value of the position taken in the bond with duration $D_1 = 3.2$ and convexity $C_1 = 16$ is $B_1 = \$1,000,000$. The value of the position taken in the bond with duration $D_2 = 4$ and convexity $C_2 = 24$ is $B_2 = \$2,500,000$.

[13] Portfolio immunization is similar, in theory, to hedging portfolios of derivative securities with respect to the Greeks; see section 3.8 for more details.

Denote by $V = B_1 + B_2$ the value of the bond portfolio. Note that $V = \$3,500,000$. The dollar duration and dollar convexity of the portfolio are

$$\begin{aligned} D_\$(V) &= D_\$(B_1) + D_\$(B_2) = B_1 D_1 + B_2 D_2 = \$13,200,000. \\ C_\$(V) &= C_\$(B_1) + C_\$(B_2) = B_1 C_1 + B_2 C_2 = \$76,000,000. \end{aligned}$$

(ii) Recall from (6.85) that the change in the value of a bond portfolio for a small parallel shift δr in the zero rate curve can be approximated as follows:

$$\Delta V \approx - D_\$(V)\delta r + \frac{C_\$(V)}{2}(\delta r)^2. \tag{6.91}$$

For $\delta r = 0.001$, we find from (6.91) that $\Delta V \approx -\$13,162$. Therefore, the new value of the bond portfolio is

$$V_{new} = V + \Delta V \approx \$3,500,000 - \$13,162 = \$3,486,838.$$

(iii) Let B_3 and B_4 be the values of the positions taken in the bond with duration $D_3 = 1.6$ and convexity $C_3 = 12$, and in the bond with duration $D_4 = 3.2$ and convexity $C_4 = 20$, respectively.

If $\Pi = V + B_3 + B_4$ denotes the value of the new portfolio, then

$$\begin{aligned} D_\$(\Pi) &= D_\$(V) + D_\$(B_3) + D_\$(B_4) \\ &= \$13.2\text{mil} + D_3 B_3 + D_4 B_4 \\ &= \$13.2\text{mil} + 1.6 B_3 + 3.2 B_4; \\ C_\$(\Pi) &= D_\$(V) + D_\$(B_3) + D_\$(B_4) \\ &= \$76\text{mil} + C_3 B_3 + C_4 B_4; \\ &= \$76\text{mil} + 12 B_3 + 20 B_4. \end{aligned}$$

Then, $D_\$(\Pi) = 0$ and $C_\$(\Pi) = 0$ if and only if

$$\begin{cases} \$13.2\text{mil} + 1.6 B_3 + 3.2 B_4 = 0; \\ \$76\text{mil} + 12 B_3 + 20 B_4 = 0, \end{cases} \tag{6.92}$$

The solution of the system (6.92) is $B_3 = \$3.25\text{mil}$ and $B_4 = -\$5.75\text{mil}$.

We conclude that, to immunize the portfolio, one should sell short \$5.75 million worth of the bond with duration 3.2 and convexity 20 and use the proceeds to buy \$3.25 million worth of the bond with duration 1.6 and convexity 12. The immunized portfolio will consist of the following positions:

- a long position worth \$1 million in bond 1;
- a long position worth \$2.5 million in bond 2;
- a long position worth \$3.25 million in bond 3;
- a short position worth \$5.75 million in bond 4;
- a cash position of \$2.5 million. □