6.11 Exercises

1. Show that the cubic Taylor approximation of $\sqrt{1+x}$ around 0 is

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

2. Use the Taylor series expansion of the function e^x to find the value of $e^{0.25}$ with six decimal digits accuracy.

3. Show that

$$e^{-x} - \frac{1}{1+x} = O(x^2), \quad \text{as } x \to 0.$$

4. (i) Let $g(x)$ be an infinitely differentiable function. Find the linear and quadratic Taylor approximations of $e^{g(x)}$ around the point 0.

   (ii) Use the result above to compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$.

   (iii) Compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$ by using Taylor approximations of e^x and e^{x^2}.

5. Find the Taylor series expansion of the functions

$$\ln(1-x^2) \quad \text{and} \quad \frac{1}{1-x^2}$$

   around the point 0, using the Taylor series expansions (6.48) and (6.49) of $\ln(1-x)$ and $\frac{1}{1-x}$ around 0.

6. Prove that

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}-\frac{1}{12x}} < e < \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}, \quad \forall\, x \geq 1.$$

   Use the fact that $\left(1+\frac{1}{x}\right)^x < e < \left(1+\frac{1}{x}\right)^{x+1}$, for all $x \geq 1$.

7. Compute the Taylor series expansion of

$$\ln\left(\frac{1+x}{1-x}\right)$$

   around the point 0, and find its radius of convergence.

8. (i) Find the radius of convergence of the series

$$1 + \frac{x^4}{2!} + \frac{x^8}{4!} + \frac{x^{12}}{6!} + \ldots \tag{6.127}$$

(ii) Show that the series from (6.127) is the Taylor series expansion of the function

$$\frac{e^{x^2} + e^{-x^2}}{2}.$$

9. Let

$$T(x) \; = \; \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \, x^k}{k}$$

be the Taylor series expansion of $f(x) = \ln(1+x)$. In section 6.3.1, we showed that $T(x) = f(x)$ if $|x| \leq \frac{1}{2}$. In this exercise, we show that $T(x) = f(x)$ for all x such that $|x| < 1$.

Let $P_n(x)$ be the Taylor polynomial of degree n corresponding to $f(x)$. Since $T(x) = \lim_{n\to\infty} P_n(x)$, it follows that $f(x) = T(x)$ for all $|x| < 1$ if and only if

$$\lim_{n\to\infty} |f(x) - P_n(x)| \; = \; 0, \quad \forall \; |x| < 1. \tag{6.128}$$

(i) Use (6.51) and the integral formula (6.4) for the Taylor approximation error to show that, for any x,

$$f(x) - P_n(x) \; = \; \int_0^x \frac{(-1)^{n+2} \, (x-t)^n}{(1+t)^{n+1}} \, dt.$$

(ii) Show that, for any $0 \leq x < 1$,

$$|f(x) - P_n(x)| \; \leq \; \int_0^x \left(\frac{x-t}{1+t}\right)^n \frac{1}{1+t} \, dt \; \leq \; x^n \ln(1+x). \tag{6.129}$$

Use (6.129) to prove that (6.128) holds for all x such that $0 \leq x < 1$.

(iii) Assume that $-1 < x \leq 0$. Let $s = -x$. Show that

$$|f(x) - P_n(x)| \; = \; \int_0^s \frac{(s-z)^n}{(1-z)^{n+1}} \, dz.$$

Note that $\frac{s-z}{1-z} \leq s$, for all $0 \leq z \leq s < 1$, and obtain that

$$|f(x) - P_n(x)| \; \leq \; s^n |\ln(1-s)| \; = \; (-x)^n |\ln(1+x)|.$$

Conclude that (6.128) holds true for all x such that $-1 < x \leq 0$.

10. The goal of this exercise is to compute

$$\int_0^1 \ln(1-x)\ln(x)\ dx. \tag{6.130}$$

(i) Show that

$$\lim_{x \searrow 0} (\ln(1-x)\ln(x)) \;=\; \lim_{x \nearrow 1} (\ln(1-x)\ln(x)) \;=\; 0,$$

and conclude that the integral (6.130) can be regarded as a definite integral.

(ii) Use the Taylor series expansion of $\ln(1-x)$ for $|x| < 1$ to show that

$$\int_0^1 \ln(1-x)\ln(x)\ dx \;=\; -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln(x)\ dx.$$

(iii) Prove that

$$\int_0^1 \ln(1-x)\ln(x)\ dx \;=\; \sum_{k=1}^{\infty} \frac{1}{n(n+1)^2}.$$

(iv) Use that fact that

$$\sum_{k=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

to obtain that

$$\int_0^1 \ln(1-x)\ln(x)\ dx \;=\; 2 - \frac{\pi^2}{6}.$$

11. In the Cox-Ross-Rubinstein parametrization for a binomial tree, the up and down factors u and d, and the risk–neutral probability p of the price going up during one time step are

$$u \;=\; A + \sqrt{A^2 - 1}; \tag{6.131}$$

$$d \;=\; A - \sqrt{A^2 - 1}; \tag{6.132}$$

$$p \;=\; \frac{e^{r\delta t} - d}{u - d}, \tag{6.133}$$

where

$$A \;=\; \frac{1}{2}\left(e^{-r\delta t} + e^{(r+\sigma^2)\delta t}\right).$$

Use Taylor expansions to show that, for a small time step δt, u, d and p may be approximated by

$$u = e^{\sigma\sqrt{\delta t}}; \tag{6.134}$$
$$d = e^{-\sigma\sqrt{\delta t}}; \tag{6.135}$$
$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\delta t}. \tag{6.136}$$

In other words, write the Taylor expansions for (6.131–6.133) and for (6.134–6.136) and show that they are identical if all the terms of order $O(\delta t)$ and smaller are neglected.

12. (i) What is the approximate value $P_{approx,r=0,q=0}$ of an at-the-money put option on a non–dividend–paying underlying asset with spot price $S = 60$, volatility $\sigma = 0.25$, and maturity $T = 1$ year, if the constant risk–free interest rate is $r = 0$?

    (ii) Compute the Black–Scholes value $P_{BS,r=0,q=0}$ of the put option, and estimate the relative approximate error

    $$\frac{|P_{BS,r=0,q=0} - P_{approx,r=0,q=0}|}{P_{BS,r=0,q=0}}.$$

    (iii) Assume that $r = 0.06$ and $q = 0.03$. Use formula (6.103) to compute $P_{approx,r=0.06,q=0.03}$, and estimate the relative approximate error

    $$\frac{|P_{BS,r=0.06,q=0.03} - P_{approx,r=0.06,q=0.03}|}{P_{BS,r=0.06,q=0.03}}, \tag{6.137}$$

    where $P_{BS,r=0.06,q=0.03}$ is the Black–Scholes value of the put option.

13. It is interesting to note that the approximate formulas (6.102) and (6.103) for ATM call and put options do not satisfy the Put–Call parity:

    $$P + Se^{-qT} - C = S\left(e^{-qT} - (r-q)T\right) \neq S\,e^{-rT} = K\,e^{-rT}.$$

    Based on the linear Taylor expansion $e^{-x} \approx 1 - x$, the formulas (6.102) and (6.103) can be modified to accommodate the Put–Call parity, by replacing rT and qT by $1 - e^{-rT}$ and $1 - e^{-qT}$, respectively. The resulting formulas are

    $$C \approx \sigma S\sqrt{\frac{T}{2\pi}}\,\frac{e^{-qT} + e^{-rT}}{2} + \frac{S\left(e^{-qT} - e^{-rT}\right)}{2}; \tag{6.138}$$
    $$P \approx \sigma S\sqrt{\frac{T}{2\pi}}\,\frac{e^{-qT} + e^{-rT}}{2} - \frac{S\left(e^{-qT} - e^{-rT}\right)}{2}. \tag{6.139}$$

(i) Show that the Put–Call parity is satisfied by the approximations (6.138) and (6.139).

(ii) Estimate how good the new approximation (6.139) is, for an ATM put with $S = 60$, $q = 0.03$, $\sigma = 0.25$, and $T = 1$, if $r = 0.06$, by computing the corresponding relative approximate error. Compare this error with the relative approximate error (6.137) found in the previous exercise, when the approximation (6.103) was used.

14. Consider an ATM put option with strike 40 on a non–dividend paying asset with volatility 30%, and assume zero interest rates.

    Compute the relative approximation error of the approximation $P \approx \sigma S\sqrt{\frac{T}{2\pi}}$ if the put option expires in 3, 5, 10, and 20 years.

15. Consider an ATM put option with strike 40 on an asset with volatility 30% and paying 2% dividends continuously. Assume that the interest rates are constant at 4.5%. Compute the relative approximation error to the Black–Scholes value of the option of the approximate value

$$P_{approx,r\neq 0,q\neq 0} \;=\; \sigma S\sqrt{\frac{T}{2\pi}}\left(1-\frac{(r+q)T}{2}\right) \;-\; \frac{(r-q)T}{2}S,$$

    if the put option expires in 1, 3, 5, 10, and 20 years.

16. A five year bond worth 101 has duration 1.5 years and convexity equal to 2.5. Use both formula (2.57), which does not include any convexity adjustment, and formula (6.73) to find the price of the bond if the yield increases by ten basis points (i.e., 0.001), fifty basis points, one percent, and two percent, respectively.

17. Consider a bond portfolio worth \$50mil with DV01 equal to \$10,000 and dollar convexity equal to \$400mil.

    (i) Assume that the yield curve moves up by fifty basis points. What is the new value of your bond portfolio?

    (ii) The following bonds are available for trading:

	Principal	Value	Duration	Convexity
Bond 1	1000	1100	2.5	12
Bond 2	100	106	3	9

    How do you immunize the portfolio?

18. (i) If the current zero rate curve is

$$r_1(0,t) \;=\; 0.025 + \frac{1}{100}\exp\left(-\frac{t}{100}\right) + \frac{t}{100(t+1)},$$

find the yield of a four year semiannual coupon bond with coupon rate 6%. Assume that interest is compounded continuously and that the face value of the bond is 100.

(ii) If the zero rates have a parallel shift up by 10, 20, 50, 100, and 200 basis points, respectively, i.e., if the zero rate curve changes from $r_1(0,t)$ to $r_2(0,t) = r_1(0,t) + dr$, with $dr = \{0.001, 0.002, 0.005, 0.01, 0.02\}$, find out by how much does the yield of the bond increase in each case.

Note: A small parallel shift in the zero rate curves results in a shift of similar size and direction for the yield of most bonds; see section 6.5 for details. This result will be tested for the bond considered here for parallel shifts ranging from small shifts (ten basis points) to large shifts (200 basis points).