Using (8.31) and the definition (8.30) for $\Theta(C)$, we conclude that formula (8.29) is mathematically correct:

$$\Theta(C) \;=\; -\frac{\partial C}{\partial(T-t)} \;=\; -\frac{\partial C}{\partial T}.$$

We emphasize again that while formula (8.29) holds true, the financially insightful formula to use is (8.30), which shows that the Theta of the option is equal to the negative rate of change of the value of the option with respect to the time left until maturity.

8.6 Integrating the density function of the standard normal variable

Recall from section 3.3 that the probability density function of the standard normal variable is

$$f(t) \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}.$$

We want to show that $f(t)$ is indeed a density function. It is clear that $f(t) \geq 0$ for any $t \in \mathbb{R}$. According to (3.39), we also have to prove that

$$\int_{-\infty}^{\infty} f(t)\, dt \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt \;=\; 1. \tag{8.32}$$

We use the substitution $t = \sqrt{2}x$. Then $dt = \sqrt{2}dx$, and $t = -\infty$ and $t = \infty$ are mapped into $x = -\infty$ and $x = \infty$, respectively. Therefore,

$$\int_{-\infty}^{\infty} f(t)\, dt \;=\; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

Thus, in order to prove (8.32), we only need to show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}. \tag{8.33}$$

Rather surprisingly, the polar coordinates change of variables can be used to prove (8.33). Let

$$I \;=\; \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

We want to show that

$$I \;=\; \sqrt{\pi}.$$

Since x is just an integrating variable, we can also write the integral I in terms of another integrating variable, denoted by y, as follows:

$$I = \int_{-\infty}^{\infty} e^{-y^2}\ dy.$$

Then,

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\ dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\ dxdy \\ &= \int\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dxdy. \end{aligned}$$

(Note that, while the second equality may be intuitively clear, a result similar to Theorem 8.1 is required for a rigorous derivation.)

We use the polar coordinates transformation (8.15) to evaluate the last integral. We change the variables $(x, y) \in \mathbb{R}^2$ to $(r, \theta) \in [0,\ \infty) \times [0,\ 2\pi)$ given by $x = r\cos\theta$ and $y = r\sin\theta$. From (8.16), we find that

$$\begin{aligned} I^2 &= \int\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\ dxdy \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-\left((r\cos\theta)^2+(r\sin\theta)^2\right)}\ d\theta dr \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-\left(r^2\ (\cos^2\theta+\sin^2\theta)\right)}\ d\theta dr \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-r^2}\ d\theta dr \\ &= \int_0^{\infty} 2\pi\ r\ e^{-r^2}\ dr \\ &= 2\pi\ \lim_{t\to\infty}\int_0^{t} r\ e^{-r^2}\ dr \\ &= 2\pi\ \lim_{t\to\infty}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^t \\ &= \pi. \end{aligned}$$

Thus, we proved that $I^2 = \pi$ and therefore that $I = \sqrt{\pi}$, i.e.,

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = I = \sqrt{\pi}.$$

As shown above, this is equivalent to showing that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\ dt = 1,$$