3.6.1 Explaining the magic of Greeks computations

It is interesting to note that the formulas (3.72–3.81) for the Greeks are simpler than expected. For example, the Delta of a call option is defined as

$$\Delta(C) = \frac{\partial C}{\partial S}.$$

Differentiating the Black–Scholes formula (3.59) with respect to S, we obtain

$$\begin{aligned}\Delta(C) &= e^{-q(T-t)}N(d_1)\\ &\quad + Se^{-q(T-t)}\frac{\partial}{\partial S}(N(d_1)) - Ke^{-r(T-t)}\frac{\partial}{\partial S}(N(d_2)),\end{aligned} \tag{3.83}$$

since both d_1 and d_2 are functions of S; cf. (3.61) and (3.62).

However, we know from (3.72) that

$$\Delta(C) = e^{-q(T-t)}N(d_1). \tag{3.84}$$

To understand how (3.83) reduces to (3.84), we apply chain rule and obtain that

$$\frac{\partial}{\partial S}(N(d_1)) = N'(d_1)\frac{\partial d_1}{\partial S}; \tag{3.85}$$

$$\frac{\partial}{\partial S}(N(d_2)) = N'(d_2)\frac{\partial d_2}{\partial S}. \tag{3.86}$$

Then, using (3.85) and (3.86), we can write (3.83) as

$$\begin{aligned}\Delta(C) &= e^{-q(T-t)}N(d_1)\\ &\quad + Se^{-q(T-t)}\,N'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\,N'(d_2)\frac{\partial d_2}{\partial S}.\end{aligned} \tag{3.87}$$

Note that the formulas (3.61) and (3.62) for d_1 and d_2 can be written as

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}; \tag{3.88}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S}{K}\right) + (r-q)(T-t)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}. \tag{3.89}$$

The following result explains why (3.87) reduces to (3.84):

Lemma 3.15. *Let d_1 and d_2 be given by (3.88) and (3.89). Then*

$$Se^{-q(T-t)}\,N'(d_1) = Ke^{-r(T-t)}\,N'(d_2). \tag{3.90}$$

Proof. Recall that $N(z)$ is the cumulative distribution of the standard normal variable, i.e.,

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx.$$

From Lemma 2.3, we find that $N'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. Then,

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}}; \tag{3.91}$$

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{\frac{-d_2^2}{2}}. \tag{3.92}$$

Therefore, in order to prove (3.90), it is enough to show that the following formula holds true:

$$Se^{-q(T-t)} e^{-\frac{d_1^2}{2}} = Ke^{-r(T-t)} e^{-\frac{d_2^2}{2}},$$

which can also be written as

$$\frac{Se^{(r-q)(T-t)}}{K} = \exp\left(\frac{d_1^2 - d_2^2}{2}\right). \tag{3.93}$$

(Recall the notation $\exp(x) = e^x$.)

From (3.88) and (3.89), it is easy to see that

$$\begin{aligned} d_1^2 - d_2^2 &= d_1^2 - (d_1 - \sigma\sqrt{T-t})^2 = 2d_1\sigma\sqrt{T-t} - \sigma^2(T-t) \\ &= 2\left(\ln\left(\frac{S}{K}\right) + (r-q)(T-t)\right) \\ &= 2\ln\left(\frac{Se^{(r-q)(T-t)}}{K}\right). \end{aligned} \tag{3.94}$$

Formula (3.93) follows immediately from (3.94). □

We return our attention to proving formula (3.84) for $\Delta(C)$.

From (3.88) and (3.89), we find that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}}. \tag{3.95}$$

Using (3.95) and Lemma 3.15, we conclude that formula (3.87) becomes

$$\begin{aligned} \Delta(C) &= e^{-q(T-t)}N(d_1) + Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} \\ &= e^{-q(T-t)}N(d_1) + Se^{-q(T-t)}\, N'(d_1)\left(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right) \\ &= e^{-q(T-t)}N(d_1). \end{aligned}$$

Formula (3.84) is therefore proven.

The simplified formulas (3.76), (3.78), and (3.80) for the vega, Θ, and ρ of a European call option[6] are obtained similarly using Lemma 3.15.

[6]Note that the formulas (3.77), (3.79), and (3.81) for the vega, Θ, and ρ of a European put option can be obtained from (3.76), (3.78), and (3.80) by using the Put–Call parity.

The formula for vega(C):

We differentiate the Black–Scholes formula (3.59) with respect to σ. Following the same steps as in the computation for the Delta of the call option, i.e., using chain rule and Lemma 2.3, we obtain that

$$\text{vega}(C) \;=\; \frac{\partial C}{\partial \sigma} \;=\; Se^{-q(T-t)}\; N'(d_1)\; \frac{\partial d_1}{\partial \sigma} \;-\; Ke^{-r(T-t)}\; N'(d_2)\; \frac{\partial d_2}{\partial \sigma}.$$

Using the result of Lemma 3.15, we conclude that

$$\text{vega}(C) \;=\; Se^{-q(T-t)}\; N'(d_1)\; \left(\frac{\partial d_1}{\partial \sigma} \;-\; \frac{\partial d_2}{\partial \sigma} \right).$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$, we find that $d_1 - d_2 = \sigma\sqrt{T-t}$ and thus

$$\frac{\partial d_1}{\partial \sigma} \;-\; \frac{\partial d_2}{\partial \sigma} \;=\; \sqrt{T-t}.$$

Then, using the fact that $N'(d_1) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{-d_1^2}{2}}$, see (3.91), we conclude that

$$\text{vega}(C) \;=\; \frac{1}{\sqrt{2\pi}}\; Se^{-q(T-t)}\; e^{-\frac{d_1^2}{2}}\; \sqrt{T-t},$$

which is the same as formula (3.76).

The formula for $\Theta(C)$:

We differentiate the Black–Scholes formula (3.59) with respect to t. Using chain rule and Lemma 2.3 we obtain that

$$\begin{aligned} \Theta(C) \;&=\; Se^{-q(T-t)}\; N'(d_1)\; \frac{\partial d_1}{\partial t} \;+\; qSe^{-q(T-t)} N(d_1) \\ &\quad -\; Ke^{-r(T-t)}\; N'(d_2)\; \frac{\partial d_2}{\partial t} \;-\; rKe^{-r(T-t)} N(d_2). \end{aligned}$$

Using the result of Lemma 3.15, we conclude that

$$\begin{aligned} \Theta(C) \;&=\; Se^{-q(T-t)}\; N'(d_1)\; \left(\frac{\partial d_1}{\partial t} \;-\; \frac{\partial d_2}{\partial t} \right) \\ &\quad +\; qSe^{-q(T-t)} N(d_1) \;-\; rKe^{-r(T-t)} N(d_2). \end{aligned}$$

Since $d_1 - d_2 = \sigma\sqrt{T-t}$, we find that

$$\frac{\partial d_1}{\partial t} \;-\; \frac{\partial d_2}{\partial t} \;=\; -\,\frac{\sigma}{2\sqrt{T-t}}.$$

Since $N'(d_1) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{-d_1^2}{2}}$, cf. (3.91), we conclude that

$$\begin{aligned} \Theta(C) \;&=\; -\frac{1}{\sqrt{2\pi}}\; Se^{-q(T-t)}\; e^{-\frac{d_1^2}{2}}\; \frac{\sigma}{2\sqrt{T-t}} \\ &\quad +\; qSe^{-q(T-t)} N(d_1) \;-\; rKe^{-r(T-t)} N(d_2), \end{aligned}$$

which is the same as formula (3.78).

The formula for $\rho(C)$:
We differentiate the Black–Scholes formula (3.59) with respect to r. Using Chain Rule and Lemma 2.3 we obtain that

$$\rho(C) = Se^{-q(T-t)}N'(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r} + K(T-t)e^{-r(T-t)}N(d_2).$$

Using Lemma 3.15, we find that

$$\rho(C) \;=\; Se^{-q(T-t)}\;N'(d_1)\;\left(\frac{\partial d_1}{\partial r} \;-\; \frac{\partial d_2}{\partial r}\right) \;+\; K(T-t)e^{-r(T-t)}N(d_2).$$

Since $d_1 - d_2 = \sigma\sqrt{T-t}$, we find that

$$\frac{\partial d_1}{\partial r} \;=\; \frac{\partial d_2}{\partial r},$$

and therefore,

$$\rho(C) \;=\; K(T-t)e^{-r(T-t)}N(d_2),$$

which is the same as formula (3.80).

3.7 Implied volatility

The only parameter needed in the Black–Scholes formulas (3.59–3.62) which is not directly observable in the markets is the volatility σ of the underlying asset. The risk free rate r and the continuous dividend yield q of the asset can be estimated from market data; the maturity date T and the strike K of the option, as well as the spot price S of the underlying asset are known when a price for the option is quoted.

Definition 3.7. *The implied volatility σ_{imp} is the value of the volatility parameter σ that makes the Black–Scholes value of the option equal to the traded price of the option.*

Denote by $C_{BS}(S, K, T, \sigma, r, q)$ the Black–Scholes value of a call option with strike K and maturity T on an underlying asset with spot price S paying dividends continuously at the rate q, if interest rates are constant and equal to r. Let C be the market price of a call with parameters S, K, T, r, and q, the implied volatility σ_{imp} corresponding to the price C is, by definition, the solution to

$$C_{BS}(S, K, T, \sigma_{imp}, r, q) \;=\; C. \tag{3.96}$$

The implied volatility can also be derived from the market price of a put option price P by solving

$$P_{BS}(S, K, T, \sigma_{imp}, r, q) \;=\; P, \tag{3.97}$$