6.5 Parallel shifts in the yield curve

A frequent assumption made when analyzing bond portfolios is that small, parallel shifts in the zero rate curve $r(0,t)$ result in identical changes in the yield of every bond in the portfolio.

To clarify why it is possible to make this assumption, consider a bond with future cash flows c_i at times t_i, $i = 1 : n$. Let B be the value of the bond and let y be the yield of the bond. If $r(0,t)$ is the continuously compounded zero rate curve, then, from (2.49) and (2.51), it follows that

$$B = \sum_{i=1}^{n} c_i e^{-yt_i} = \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i}. \tag{6.63}$$

If a parallel shift of the zero rate curve occurs, then the zero rate curve $r(0,t)$ changes to $r_1(0,t) = r(0,t) + \delta r$, where δr is small,[8] but not necessarily positive. Denote by B_1 and y_1 the new value and the new yield of the bond, respectively. As in (6.63), we find that

$$B_1 = \sum_{i=1}^{n} c_i e^{-y_1 t_i} = \sum_{i=1}^{n} c_i e^{-r_1(0,t_i)t_i}. \tag{6.64}$$

Denote by $\Delta y = y_1 - y$ the shift in the yield of the bond. We will show that, for δr small, it is reasonable to approximate Δy by δr.

From the linear Taylor approximation $e^x \approx 1 + x$, see (6.10), it follows that

$$e^{-(\Delta y)t_i} - 1 \approx (1 - (\Delta y)t_i) - 1 = -(\Delta y)t_i; \tag{6.65}$$

$$e^{-(\delta r)t_i} - 1 \approx (1 - (\delta r)t_i) - 1 = -(\delta r)t_i. \tag{6.66}$$

Recall that $y_1 = y + \Delta y$ and $r_1(0,t) = r(0,t) + \delta r$. From (6.63–6.66), we obtain that

$$\begin{aligned} B_1 - B &= \sum_{i=1}^{n} c_i e^{-y_1 t_i} - \sum_{i=1}^{n} c_i e^{-yt_i} \\ &= \sum_{i=1}^{n} c_i \left(e^{-y_1 t_i} - e^{-yt_i}\right) \\ &= \sum_{i=1}^{n} c_i \left(e^{-(y+\Delta y)t_i} - e^{-yt_i}\right) \\ &= \sum_{i=1}^{n} c_i e^{-yt_i} \left(e^{-(\Delta y)t_i} - 1\right) \end{aligned}$$

[8]Typical values for δr that are considered to be small are a few basis points. (One percentage point is equal to 100 basis points (bps), i.e., one basis point is equal to 0.01%.) For example, $\delta r = 10\text{bps} = 0.1\% = 0.001$ is small enough for the approximations herein.

$$\begin{aligned} &\approx \sum_{i=1}^{n} c_i e^{-yt_i} \ (-(\Delta y)t_i) \\ &= -\Delta y \sum_{i=1}^{n} t_i c_i e^{-yt_i}; \qquad (6.67) \end{aligned}$$

$$\begin{aligned} B_1 - B &= \sum_{i=1}^{n} c_i e^{-r_1(0,t_i)t_i} - \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i} \\ &= \sum_{i=1}^{n} c_i \left(e^{-r_1(0,t_i)t_i} - e^{-r(0,t_i)t_i}\right) \\ &= \sum_{i=1}^{n} c_i \left(e^{-(r(0,t)+\delta r)t_i} - e^{-r(0,t_i)t_i}\right) \\ &= \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i} \ \left(e^{-(\delta r)t_i} - 1\right) \\ &\approx \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i} \ (-(\delta r)t_i) \\ &= -\delta r \sum_{i=1}^{n} t_i c_i e^{-r(0,t_i)t_i}. \qquad (6.68) \end{aligned}$$

The following approximation is generally very accurate and states that the Macaulay duration and the Macaulay–Weill duration[9] are approximately equal:

$$\sum_{i=1}^{n} t_i c_i e^{-yt_i} \approx \sum_{i=1}^{n} t_i c_i e^{-r(0,t_i)t_i}. \qquad (6.69)$$

From (6.67–6.69), we conclude that

$$\Delta y \approx \delta r. \qquad (6.70)$$

In other words, the following result was established:

[9]The difference between the two types of duration is given by discounting with respect to the yield of the bond, for Macaulay duration, and with respect to the zero rate curve, for the Macaulay–Weill duration, i.e.,

$$\begin{aligned} D_{Mac} &= \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-yt_i}; \\ D_{Mac-Weil} &= \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-r(0,t_i)t_i}. \end{aligned}$$

Lemma 6.2. *If the zero rate curve experiences a small parallel shift of size δr, the corresponding change Δy in the yield of a bond is approximately equal to the parallel shift in the zero rates, i.e., $\Delta y \approx \delta r$.*

6.6 Connections between bond returns, duration, and convexity

Recall from section 2.7, that the modified duration and the convexity of a bond are defined as

$$D = -\frac{1}{B}\frac{\partial B}{\partial y} \quad \text{and} \quad C = \frac{1}{B}\frac{\partial^2 B}{\partial y^2}, \tag{6.71}$$

where y is the yield of a bond with value B; cf. (2.52) and (2.58).

Also, recall from (2.51) that the value B of a bond with yield y paying cash–flows c_i and time t_i, $i = 1:n$, is

$$B(y) = \sum_{i=1}^{n} c_i e^{-yt_i}. \tag{6.72}$$

If the yield of the bond changes from y to $y + \Delta y$ over a small time interval Δt, the new price of the bond can be computed as the sum of the discounted present values of all the cash flows computed using the new yield $y + \Delta y$. Since the cash flow c_i will no longer be received at time t_i, but at time $t_i - \Delta t$, we find that[10]

$$B(y + \Delta y) = \sum_{i=1}^{n} c_i e^{-(y+\Delta y)(t_i - \Delta t)}.$$

An alternative to discounting the future cash flows to compute the new price of the bond is given below.

Lemma 6.3. *Let D and C be the modified duration and the convexity of a bond with yield y and value $B = B(y)$. Then,*

$$\frac{\Delta B}{B} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2, \tag{6.73}$$

where $\Delta B = B(y + \Delta y) - B(y)$.

Note that an approximate value for the percentage return $\frac{\Delta B}{B}$ of a long bond position can be computed using formula (6.73), without requiring specific knowledge of the cash flows of the bond.

[10]We implicitly assumed that Δt is chosen small enough such that $t_1 - \Delta t > 0$, where $t_1 > 0$ is the first cash flow date.