Chapter 1

Calculus review. Options. Put–Call parity.

1.1 Exercises

1. Use integration by parts to compute
$$\int \ln(x)\ dx.$$

2. Compute
$$\int \frac{1}{x\ln(x)}\ dx.$$
Hint: Use the substitution $u = \ln(x)$.

3. Show that $(\tan x)' = 1/(\cos x)^2$ and conclude that
$$\int \frac{1}{1+x^2}\ dx \ = \ \arctan(x) \ + \ C.$$
Note: The antiderivative of a rational function is often computed using the substitution $x = \tan\left(\frac{z}{2}\right)$.

4. Compute
$$\int x^n \ln(x)\ dx.$$

5. Compute
$$\int x^n e^x\ dx.$$

6. Compute
$$\int (\ln(x))^n \, dx.$$

7. Show that
$$\left(1+\frac{1}{x}\right)^x \;<\; e \;<\; \left(1+\frac{1}{x}\right)^{x+1}, \quad \forall\, x \geq 1.$$

8. Use l'Hôpital's rule to show that the following two Taylor approximations hold when x is close to 0:
$$\sqrt{1+x} \;\approx\; 1 \;+\; \frac{x}{2};$$
$$e^x \;\approx\; 1 \;+\; x \;+\; \frac{x^2}{2}.$$
In other words, show that the following limits exist and are constant:
$$\lim_{x\to 0} \frac{\sqrt{1+x} \;-\; \left(1+\frac{x}{2}\right)}{x^2} \quad \text{and} \quad \lim_{x\to 0} \frac{e^x \;-\; \left(1+x+\frac{x^2}{2}\right)}{x^3}.$$

9. Compute the following limits:
$$\text{(i)} \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x};$$
$$\text{(ii)} \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x+2}.$$

10. Use the definition
$$e \;=\; \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$$
to show that
$$\frac{1}{e} \;=\; \lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x.$$
Hint: Use the fact that
$$\frac{1}{1+\frac{1}{x}} \;=\; \frac{x}{x+1} \;=\; 1-\frac{1}{x+1}.$$

11. Let K, T, σ and r be positive constants, and define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) \;=\; \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{y^2}{2}}\, dy,$$

where

$$b(x) \;=\; \left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T\right) / \left(\sigma\sqrt{T}\right).$$

Compute $g'(x)$.

Note: This function is related to the Delta of a plain vanilla Call option.

12. Let $f(x)$ be a continuous function. Show that

$$\lim_{h\to 0} \frac{1}{2h} \int_{a-h}^{a+h} f(x)\, dx \;=\; f(a), \quad \forall \;\; a \in \mathbb{R}.$$

Note: Let $F(x) = \int f(x)\, dx$. The central finite difference approximation of $F'(a)$ is

$$F'(a) \;=\; \frac{F(a+h) - F(a-h)}{2h} \;+\; O\left(h^2\right), \tag{1.1}$$

as $h \to 0$ (if $F^{(3)}(x) = f''(x)$ is continuous). Since $F'(a) = f(a)$, formula (1.1) can be written as

$$f(a) \;=\; \frac{1}{2h} \int_{a-h}^{a+h} f(x)\, dx \;+\; O(h^2).$$

13. Let

$$f(x) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function which is uniformly bounded, i.e., there exists a constant C such that $|g(x)| \leq C$ for all $x \in \mathbb{R}$. Then, show that

$$\lim_{\sigma\searrow 0} \int_{-\infty}^{\infty} f(x) g(x)\, dx \;=\; g(\mu).$$

Note: The function $f(x)$ is the probability density function of a normal random variable with mean μ and standard deviation σ. This exercise shows that the probability density functions of normal variables with standard deviation going to 0 converges, in the sense of distributions, to the delta function corresponding to the mean μ.

14. Let c_i and t_i, $i = 1 : n$, be positive constants.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(y) \;=\; \sum_{i=1}^{n} c_i e^{-yt_i}.$$

Compute $f'(y)$ and $f''(y)$.

(ii) Let $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(y) \;=\; \sum_{i=1}^{n} c_i \left(1 + \frac{y}{m}\right)^{-mt_i}.$$

Compute $g'(y)$ and $g''(y)$.

Note: The functions $f(y)$ and $g(y)$ represent the price of a bond with cash flows c_i paid at time t_i as a function of the yield y of the bond, if the yield is compounded continuously, and if the yield is compounded discretely m times a year, respectively.

15. Let $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) \;=\; 2x_1^2 - x_1x_2 + 3x_2x_3 - x_3^2,$$

where $x = (x_1, x_2, x_3)$.

(i) Compute the gradient and Hessian of the function $f(x)$ at the point $a = (1, -1, 0)$, i.e., compute $Df(1, -1, 0)$ and $D^2f(1, -1, 0)$.

(ii) Show that

$$f(x) \;=\; f(a) + Df(a)\;(x-a) + \frac{1}{2}\;(x-a)^t\;D^2f(a)\;(x-a). \qquad (1.2)$$

Here, x, a, and $x - a$ are 3×1 column vectors, i.e.,

$$x \;=\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad a \;=\; \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad x - a \;=\; \begin{pmatrix} x_1 - 1 \\ x_2 + 1 \\ x_3 \end{pmatrix}.$$

Note: Formula (1.2) is the quadratic Taylor approximation of $f(x)$ around the point a. Since $f(x)$ is a second order polynomial, the quadratic Taylor approximation of $f(x)$ is exact.

16. Let
$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad \text{for } t > 0,\ x \in \mathbb{R}.$$
Compute $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, and show that
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$
Note: This exercise shows that the function $u(x,t)$ is a solution of the heat equation. In fact, $u(x,t)$ is the fundamental solution of the heat equation, and is used in the PDE derivation of the Black–Scholes formula for pricing European plain vanilla options.

Also, note that $u(x,t)$ is the same as the density function of a normal variable with mean 0 and variance $2t$.

17. Show that the values of a plain vanilla put option and of a plain vanilla call option with the same maturity and strike, and on the same underlying asset, are equal if and only if the strike is equal to the forward price.

18. Consider a portfolio with the following positions:
    - long one call option with strike $K_1 = 30$;
    - short two call options with strike $K_2 = 35$;
    - long one call option with strike $K_3 = 40$.

    All options are on the same underlying asset and have maturity T. Draw the payoff diagram at maturity of the portfolio, i.e., plot the value of the portfolio $V(T)$ at maturity as a function of $S(T)$, the price of the underlying asset at time T.

    Note: This is a butterfly spread. A trader takes a long position in a butterfly spread if the price of the underlying asset at maturity is expected to be in the $K_1 \leq S(T) \leq K_3$ range.

19. Draw the payoff diagram at maturity of a bull spread with a long position in a call with strike 30 and short a call with strike 35, and of a bear spread with long a put of strike 20 and short a put of strike 15.

20. The prices of three call options with strikes 45, 50, and 55, on the same underlying asset and with the same maturity, are \$4, \$6, and \$9, respectively. Create a butterfly spread by going long a 45–call and a

55–call, and shorting two 50–calls. What are the payoff and the P&L at maturity of the butterfly spread? When would the butterfly spread be profitable? Assume, for simplicity, that interest rates are zero.

21. Which of the following two portfolios would you rather hold:

    - Portfolio 1: Long one call option with strike $K = X - 5$ and long one call option with strike $K = X + 5$;
    - Portfolio 2: Long two call options with strike $K = X$?

    (All options are on the same asset and have the same maturity.)

22. A stock with spot price \$42 pays dividends continuously at a rate of 3%. The four months put and call options with strike 40 on this asset are trading at \$2 and \$4, respectively. The risk-free rate is constant and equal to 5% for all times. Show that the Put-Call parity is not satisfied and explain how would you take advantage of this arbitrage opportunity.

23. The bid and ask prices for a six months European call option with strike 40 on a non–dividend–paying stock with spot price 42 are \$5 and \$5.5, respectively. The bid and ask prices for a six months European put option with strike 40 on the same underlying asset are \$2.75 and \$3.25, respectively. Assume that the risk free rate is equal to 0. Is there an arbitrage opportunity present?

24. Denote by C_{bid} and C_{ask}, and by P_{bid} and P_{ask}, respectively, the bid and ask prices for a plain vanilla European call and for a plain vanilla European put option, both with the same strike K and maturity T, and on the same underlying asset with spot price S and paying dividends continuously at rate q. Assume that the risk–free interest rates are constant equal to r. Find necessary and sufficient no–arbitrage conditions for C_{bid}, C_{ask}, P_{bid}, and P_{ask}.

25. You expect that an asset with spot price \$35 will trade in the \$40–\$45 range in one year. One year at–the–money calls on the asset can be bought for \$4. To act on the expected stock price appreciation, you decide to either buy the asset, or to buy ATM calls. Which strategy is better, depending on where the asset price will be in a year?