1.2 Solutions to Chapter 1 Exercises

Problem 1: Compute

$$\int \ln(x)\ dx.$$

Solution: Using integration by parts, we find that

$$\begin{aligned}\int \ln(x)\ dx &= \int (x)' \ln(x)\ dx \;=\; x\ln(x) - \int x(\ln(x))'\ dx\\ &= x\ln(x) - \int 1\ dx \;=\; x\ln(x) - x + C. \quad \square\end{aligned}$$

Problem 2: Compute

$$\int \frac{1}{x\ln(x)}\ dx$$

by using the substitution $u = \ln(x)$.

Solution: Let $u = \ln(x)$. Then $du = \frac{dx}{x}$ and therefore

$$\int \frac{1}{x\ln(x)}\ dx \;=\; \int \frac{1}{u}\ du \;=\; \ln(|u|) \;=\; \ln(|\ln(x)|) + C. \quad \square$$

Problem 3: Show that $(\tan x)' = 1/(\cos x)^2$ and conclude that

$$\int \frac{1}{1+x^2}\ dx \;=\; \arctan(x) \;+\; C.$$

Solution: Using the Quotient Rule, we find that

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \;=\; \frac{(\sin x)'\cos x - \sin x(\cos x)'}{(\cos x)^2}\\ &= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} \;=\; \frac{1}{(\cos x)^2}. \end{aligned} \tag{1.3}$$

To prove that $\int \frac{1}{1+x^2} dx = \arctan(x)$, we will show that

$$(\arctan(x))' \;=\; \frac{1}{1+x^2}.$$

Let $f(x) = \tan x$. Then $\arctan(x) = f^{-1}(x)$. Recall that

$$\left(f^{-1}(x)\right)' \;=\; \frac{1}{f'(f^{-1}(x))}$$

and note that $f'(x) = (\tan x)' = \frac{1}{(\cos x)^2}$; cf. (1.3). Therefore,

$$(\arctan(x))' = (\cos(f^{-1}(x)))^2 = (\cos(\arctan(x)))^2. \tag{1.4}$$

Let $\alpha = \arctan(x)$. Then $\tan(\alpha) = x$. It is easy to see that

$$x^2 + 1 = \frac{1}{(\cos(\alpha))^2},$$

since $(\sin(\alpha))^2 + (\cos(\alpha))^2 = 1$. Thus,

$$(\cos(\arctan(x)))^2 \;=\; (\cos(\alpha))^2 \;=\; \frac{1}{x^2+1}. \tag{1.5}$$

From (1.4) and (1.5), we conclude that

$$(\arctan(x))' \;=\; \frac{1}{x^2+1},$$

and therefore that

$$\int \frac{1}{1+x^2}\, dx \;=\; \arctan(x) \;+\; C.$$

We note that the antiderivative of a rational function is often computed using the substitution $x = \tan\left(\frac{z}{2}\right)$.

For example, to compute $\int \frac{1}{1+x^2} dx$ using the substitution $x = \tan\left(\frac{z}{2}\right)$, note that

$$dx \;=\; \frac{d}{dz}\left(\tan\left(\frac{z}{2}\right)\right) dz \;=\; \frac{1}{2(\cos\left(\frac{z}{2}\right))^2} dz.$$

Then

$$\begin{aligned}
\int \frac{1}{1+x^2}\, dx \;&=\; \int \frac{1}{1+(\tan\left(\frac{z}{2}\right))^2} \cdot \frac{1}{2(\cos\left(\frac{z}{2}\right))^2}\, dz \\
&=\; \int \frac{\left(\cos\left(\frac{z}{2}\right)\right)^2}{(\sin(\alpha))^2 + (\cos(\alpha))^2} \cdot \frac{1}{2(\cos\left(\frac{z}{2}\right))^2}\, dz \\
&=\; \int \frac{1}{2}\, dz \;=\; \frac{z}{2} = \arctan(x) \;+\; C. \quad \square
\end{aligned}$$

Problem 4: Compute

$$\int x^n \ln(x)\, dx.$$

Solution: If $n \neq -1$, we use integration by parts and find that

$$\begin{aligned}\int x^n \ln(x)\, dx &= \frac{x^{n+1}}{n+1}\ln(x) - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x}\, dx \\ &= \frac{x^{n+1}\ln(x)}{n+1} - \frac{1}{n+1}\int x^n\, dx \\ &= \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.\end{aligned}$$

For $n = -1$, we obtain that

$$\int \frac{\ln(x)}{x}\, dx = \frac{(\ln(x))^2}{2} + C. \quad \square$$

Problem 5: Compute

$$\int x^n e^x\, dx.$$

Solution: For every integer $n \geq 0$, let

$$f_n(x) = \int x^n e^x\, dx.$$

By using integration by parts, we find that

$$f_n(x) = x^n e^x - n\int x^{n-1} e^x\, dx,$$

which can be written as

$$f_n(x) = x^n e^x - n f_{n-1}(x), \quad \forall\, n \geq 1. \tag{1.6}$$

Note that[1]

$$f_0(x) = \int e^x dx = e^x.$$

By letting $n = 1$ in (1.6), we obtain that

$$f_1(x) = xe^x - f_0(x) = (x-1)e^x.$$

By letting $n = 2$ in (1.6), we obtain that

$$f_2(x) = x^2 e^x - 2f_1(x) = (x^2 - 2x + 2)e^x.$$

[1] To avoid confusions, we will not add a constant C when writing down the formulas for $f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

By letting $n = 3$ in (1.6), we obtain that

$$f_3(x) \;=\; x^3 e^x - 3f_2(x) \;=\; (x^3 - 3x^2 + 6x - 6)e^x.$$

The following general formula can be proved by induction:

$$\int x^n e^x \, dx \;=\; f_n(x) \;=\; \left(\sum_{k=0}^{n} x^k \frac{(-1)^{n-k} n!}{k!} \right) e^x \;+\; C, \quad \forall\, n \geq 0. \quad \square$$

Problem 6: Compute

$$\int (\ln(x))^n \, dx.$$

Solution: For every integer $n \geq 0$, let

$$f_n(x) \;=\; \int (\ln(x))^n \, dx.$$

By using integration by parts, it is easy to see that, for any $n \geq 1$,

$$\begin{aligned} \int (\ln(x))^n \, dx \;&=\; x(\ln(x))^n \;-\; \int x \left((\ln(x))^n \right)' \, dx \\ &=\; x(\ln(x))^n \;-\; \int x \cdot n(\ln(x))^{n-1} \cdot (\ln(x))' \, dx \\ &=\; x(\ln(x))^n \;-\; \int x \cdot n(\ln(x))^{n-1} \cdot \frac{1}{x} \, dx \\ &=\; x(\ln(x))^n - n \int (\ln(x))^{n-1} \, dx, \end{aligned}$$

and therefore

$$f_n(x) \;=\; x(\ln(x))^n - n f_{n-1}(x), \quad \forall\, n \geq 1. \tag{1.7}$$

Note that

$$f_0(x) \;=\; \int 1 dx \;=\; x.$$

By letting $n = 1$ in (1.7), we obtain that

$$f_1(x) \;=\; x\ln(x) - f_0(x) \;=\; x(\ln(x) - 1).$$

By letting $n = 2$ in (1.7), we obtain that

$$f_2(x) \;=\; x(\ln(x))^2 - 2f_1(x) \;=\; x\left((\ln(x))^2 - 2\ln(x) + 2 \right).$$

By letting $n = 3$ in (1.7), we obtain that

$$f_3(x) \;=\; x(\ln(x))^3 - 3f_2(x) \;=\; x\left((\ln(x))^3 - 3(\ln(x))^2 + 6\ln(x) - 6\right).$$

The following general formula can be obtained by induction:

$$\int (\ln(x))^n \, dx \;=\; x\sum_{k=0}^{n} \frac{(-1)^{n-k} n!}{k!} (\ln(x))^k \;+\; C, \quad \forall\, n \geq 0. \quad \Box$$

Problem 7: Show that

$$\left(1 + \frac{1}{x}\right)^x \;<\; e \;<\; \left(1 + \frac{1}{x}\right)^{x+1}, \quad \forall\, x \geq 1. \tag{1.8}$$

Solution: By taking natural logs on both sides of (1.8), we find that (1.8) is equivalent to

$$x \ln\left(1 + \frac{1}{x}\right) \;<\; 1 \;<\; (x+1)\ln\left(1 + \frac{1}{x}\right),$$

which can be written as

$$\frac{1}{x+1} \;<\; \ln\left(1 + \frac{1}{x}\right) \;<\; \frac{1}{x}, \quad \forall\, x \geq 1. \tag{1.9}$$

Let

$$f(x) \;=\; \frac{1}{x} - \ln\left(1 + \frac{1}{x}\right); \quad g(x) \;=\; \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}.$$

Then,

$$f'(x) \;=\; -\frac{1}{x^2} + \frac{1}{x(x+1)} \;=\; -\,\frac{1}{x^2(x+1)} \;<\; 0;$$

$$g'(x) \;=\; -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} \;=\; -\,\frac{1}{x(x+1)^2} \;<\; 0$$

We conclude that both $f(x)$ and $g(x)$ are decreasing functions. Since

$$\lim_{x\to\infty} f(x) \;=\; \lim_{x\to\infty} g(x) \;=\; 0,$$

it follows that $f(x) > 0$ and $g(x) > 0$ for all $x > 0$, and therefore

$$\frac{1}{x} \;>\; \ln\left(1 + \frac{1}{x}\right) \;>\; \frac{1}{x+1}, \quad \forall\, x > 0,$$

which is what we wanted to show; cf. (1.9). □

Problem 8: Use l'Hôpital's rule to show that the following two Taylor approximations hold when x is close to 0:

$$\sqrt{1+x} \approx 1 + \frac{x}{2};$$
$$e^x \approx 1 + x + \frac{x^2}{2}.$$

In other words, show that the following limits exist and are constant:

$$\lim_{x\to 0} \frac{\sqrt{1+x} - \left(1+\frac{x}{2}\right)}{x^2} \quad \text{and} \quad \lim_{x\to 0} \frac{e^x - \left(1+x+\frac{x^2}{2}\right)}{x^3}.$$

Solution: The numerator and denominator of each limit are differentiated until a finite limit is computed. L'Hôpital's rule can then be applied sequentially to obtain the value of the initial limit:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+x} - \left(1+\frac{x}{2}\right)}{x^2} &= \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}} - \frac{1}{2}}{2x} \\
&= \lim_{x\to 0} \frac{-\frac{1}{4(1+x)^{3/2}}}{2} \\
&= -\frac{1}{8}.
\end{aligned}$$

We conclude that

$$\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2), \quad \text{as} \quad x \to 0.$$

Similarly,

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - \left(1+x+\frac{x^2}{2}\right)}{x^3} &= \lim_{x\to 0} \frac{e^x - (1+x)}{3x^2} \\
&= \lim_{x\to 0} \frac{e^x - 1}{6x} \\
&= \lim_{x\to 0} \frac{e^x}{6} \\
&= \frac{1}{6},
\end{aligned}$$

and therefore

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3) \quad \text{as} \quad x \to 0. \quad \square$$

Problem 9: Compute the following limits:

$$\text{(i)} \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x};$$

$$\text{(ii)} \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x+2}.$$

Solution: (i) By multiplying the denominator of the fraction with its conjugate, it is easy to see that

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x} &= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{(\sqrt{x^2-4x+1}+x)(\sqrt{x^2-4x+1}-x)} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{(\sqrt{x^2-4x+1})^2 \ - \ x^2} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{x^2-4x+1 \ - \ x^2} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{-4x+1} \\
&= \lim_{x\to\infty} \frac{\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}+1}{-4+\frac{1}{x}} \\
&= \frac{1+1}{-4} \\
&= -\frac{1}{2}.
\end{aligned}$$

(ii) Using a similar method as before, we find that

$$\begin{aligned}
& \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x+2} \\
= & \lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-(x-2)} \\
= & \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+(x-2)}{(\sqrt{x^2-4x+1}+(x-2))(\sqrt{x^2-4x+1}-(x-2))} \\
= & \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+(x-2)}{(\sqrt{x^2-4x+1})^2 \ - \ (x-2)^2} \\
= & \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+(x-2)}{x^2-4x+1 \ - \ (x^2-4x+4)} \\
= & \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+(x-2)}{-3}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{3}\lim_{x\to\infty}\left(\sqrt{x^2-4x+1}+x-2\right) \\
&= -\infty. \quad \square
\end{aligned}
$$

Problem 10: Use the definition

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

to show that

$$\frac{1}{e} = \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x.$$

Solution: Note that

$$1-\frac{1}{x} = \frac{x-1}{x} = \frac{1}{\frac{x}{x-1}} = \frac{1}{1+\frac{1}{x-1}}.$$

Then,

$$
\begin{aligned}
\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x &= \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x-1}\right)^x} \\
&= \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x-1}\right)^{x-1}}\cdot\frac{1}{1+\frac{1}{x-1}} = \frac{1}{e},
\end{aligned}
$$

since

$$\lim_{x\to\infty} 1+\frac{1}{x-1} = 1$$

and

$$\lim_{x\to\infty}\left(1+\frac{1}{x-1}\right)^{x-1} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e. \quad \square$$

Problem 11: Let K, T, σ and r be positive constants, and define the function $g:\mathbb{R}\to\mathbb{R}$ as

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_0^{b(x)} e^{-\frac{y^2}{2}}\,dy,$$

where

$$b(x) = \left(\ln\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right)\Big/\left(\sigma\sqrt{T}\right). \tag{1.10}$$

Compute $g'(x)$.

Solution: Recall that

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(y)\, dy\right) = f(b(x))b'(x) - f(a(x))a'(x), \tag{1.11}$$

and note that

$$b'(x) = \frac{1}{x\sigma\sqrt{T}}.$$

Using (1.11) for $a(x) = 0$, $b(x)$ given by (1.10), and $f(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$, we obtain that

$$\begin{aligned} g'(x) &= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(b(x))^2}{2}} \cdot b'(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(b(x))^2}{2}\right) \frac{1}{x\sigma\sqrt{T}} \\ &= \frac{1}{x\sigma\sqrt{2\pi T}} \exp\left(-\frac{\left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T}\right). \quad \square \end{aligned}$$

Problem 12: Let $f(x)$ be a continuous function. Show that

$$\lim_{h\to 0} \frac{1}{2h} \int_{a-h}^{a+h} f(x)\, dx = f(a), \quad \forall\, a \in \mathbb{R}.$$

Solution: Let $F(x) = \int f(x)\, dx$ be the antiderivative of $f(x)$. From the Fundamental Theorem of Calculus, it follows that

$$\frac{1}{2h}\int_{a-h}^{a+h} f(x)\, dx = \frac{F(a+h) - F(a-h)}{2h}.$$

Using l'Hôpital's rule and the fact that $F'(x) = f(x)$, we find that

$$\begin{aligned} \lim_{h\to 0} \frac{1}{2h}\int_{a-h}^{a+h} f(x)\, dx &= \lim_{h\to 0} \frac{F(a+h) - F(a-h)}{2h} \\ &= \lim_{h\to 0} \frac{f(a+h) + f(a-h)}{2} \\ &= f(a), \end{aligned}$$

since $f(x)$ is a continuous function. $\square$

Problem 13: Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Assume that $g : \mathbb{R} \to \mathbb{R}$ is a uniformly bounded[2] continuous function, i.e., assume that there exists a constant C such that $|g(x)| \le C$ for all $x \in \mathbb{R}$. Show that

$$\lim_{\sigma \searrow 0} \int_{-\infty}^{\infty} f(x) g(x)\ dx \ = \ g(\mu).$$

Solution: Using the change of variables $y = \frac{x-\mu}{\sigma}$, we find that

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) g(x)\ dx \ &= \ \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)\ dx \\ &= \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\mu + \sigma y)\ e^{-\frac{y^2}{2}}\ dy. \end{aligned} \tag{1.12}$$

Recall that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\ dy \ = \ 1, \tag{1.13}$$

since, e.g., the function $\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ is the probability density function of the standard normal variable. From (1.12) and (1.13) we obtain that

$$g(\mu) - \int_{-\infty}^{\infty} f(x) g(x)\ dx \ = \ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (g(\mu) - g(\mu + \sigma y))\ e^{-\frac{y^2}{2}}\ dy. \tag{1.14}$$

Our goal is to show that the right hand side of (1.14) goes to 0 as $\sigma \searrow 0$.

Since $g(x)$ is a continuous function, it follows that, for any $\epsilon > 0$, there exists $\delta_1(\epsilon) > 0$ such that

$$|g(\mu) - g(x)| < \epsilon, \ \ \forall\ |x - \mu| < \delta_1(\epsilon). \tag{1.15}$$

Using the fact that the integral (1.13) exists and is finite, we obtain that, for any $\epsilon > 0$, there exists $\delta_2(\epsilon) > 0$ such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\delta_2(\epsilon)} e^{-\frac{y^2}{2}}\ dy \ + \ \frac{1}{\sqrt{2\pi}} \int_{\delta_2(\epsilon)}^{\infty} e^{-\frac{y^2}{2}}\ dy \ < \ \epsilon. \tag{1.16}$$

Since $|g(x)| \le C$ for all $x \in \mathbb{R}$, it follows from (1.16) that

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\delta_2(\epsilon)} |g(\mu) - g(\mu + \sigma y)|\ e^{-\frac{y^2}{2}}\ dy \\ &+ \frac{1}{\sqrt{2\pi}} \int_{\delta_2(\epsilon)}^{\infty} |g(\mu) - g(\mu + \sigma y)|\ e^{-\frac{y^2}{2}}\ dy \ < \ 2C\epsilon. \end{aligned} \tag{1.17}$$

[2]The uniform boundedness condition was chosen for simplicity, and it can be relaxed, e.g., to functions which have polynomial growth at infinity.

It is easy to see that, if $\sigma < \frac{\delta_1(\epsilon)}{\delta_2(\epsilon)}$, then

$$|(\mu + \sigma y) - \mu| = \sigma|y| < \delta_1(\epsilon)\frac{|y|}{\delta_2(\epsilon)} \leq \delta_1(\epsilon), \quad \forall\, y \in [-\delta_2(\epsilon), \delta_2(\epsilon)]. \qquad (1.18)$$

Then, from (1.15) and (1.18) we find that

$$|g(\mu) - g(\mu + \sigma y)| \;<\; \epsilon, \quad \forall\, y \in [-\delta_2(\epsilon), \delta_2(\epsilon)], \qquad (1.19)$$

and therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\delta_2(\epsilon)}^{\delta_2(\epsilon)} |g(\mu) - g(\mu + \sigma y)|\; e^{-\frac{y^2}{2}}\, dy \;<\; \epsilon. \qquad (1.20)$$

From (1.14), (1.17), and (1.20), it follows that, for any $\epsilon > 0$, there exist $\delta_1(\epsilon) > 0$ and $\delta_2(\epsilon) > 0$ such that, if $\sigma < \frac{\delta_1(\epsilon)}{\delta_2(\epsilon)}$, then

$$\begin{aligned} \left| g(\mu) - \int_{-\infty}^{\infty} f(x)g(x)\, dx \right| \;&\leq\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |g(\mu) - g(\mu + \sigma y)|\; e^{-\frac{y^2}{2}}\, dy \\ &<\; (2C + 1)\epsilon. \end{aligned}$$

We conclude, by definition, that

$$\lim_{\sigma \searrow 0} \int_{-\infty}^{\infty} f(x)g(x)\, dx \;=\; g(\mu). \quad \square$$

Problem 14: Let c_i and t_i, $i = 1 : n$, be positive constants.
(i) Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(y) \;=\; \sum_{i=1}^{n} c_i e^{-yt_i}.$$

Compute $f'(y)$ and $f''(y)$.
(ii) Let $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(y) \;=\; \sum_{i=1}^{n} c_i \left(1 + \frac{y}{m}\right)^{-mt_i}.$$

Compute $g'(y)$ and $g''(y)$.

Solution: (i) Note that

$$\begin{aligned} \left(e^{-yt_i}\right)' \;&=\; \frac{d}{dy}\left(e^{-yt_i}\right) \;=\; -\,t_i e^{-yt_i}; \\ \left(e^{-yt_i}\right)'' \;&=\; \frac{d}{dy}\left(-t_i e^{-yt_i}\right) \;=\; t_i^2 e^{-yt_i}. \end{aligned}$$

Then,

$$
\begin{aligned}
f'(y) &= -\sum_{i=1}^{n} c_i t_i e^{-yt_i}; \\
f''(y) &= \sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}.
\end{aligned}
$$

(ii) Using Chain Rule, we obtain that

$$
\begin{aligned}
\left(\left(1+\frac{y}{m}\right)^{-mt_i}\right)' &= \left(1+\frac{y}{m}\right)^{-mt_i-1} \cdot (-mt_i) \cdot \frac{1}{m} \\
&= -t_i \left(1+\frac{y}{m}\right)^{-mt_i-1}; \\
\left(\left(1+\frac{y}{m}\right)^{-mt_i}\right)'' &= -t_i \left(1+\frac{y}{m}\right)^{-mt_i-2} \cdot (-mt_i-1) \cdot \frac{1}{m} \\
&= t_i \left(t_i + \frac{1}{m}\right) \left(1+\frac{y}{m}\right)^{-mt_i-2}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
g'(y) &= -\sum_{i=1}^{n} c_i t_i \left(1+\frac{y}{m}\right)^{-mt_i-1}; \\
g''(y) &= \sum_{i=1}^{n} c_i t_i \left(t_i + \frac{1}{m}\right) \left(1+\frac{y}{m}\right)^{-mt_i-2}. \quad \Box
\end{aligned}
$$

Problem 15: Let $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$
f(x) = 2x_1^2 - x_1x_2 + 3x_2x_3 - x_3^2,
$$

where $x = (x_1, x_2, x_3)$.

(i) Compute the gradient and Hessian of the function $f(x)$ at the point $a = (1, -1, 0)$, i.e., compute $Df(1, -1, 0)$ and $D^2 f(1, -1, 0)$.

(ii) Show that

$$
f(x) = f(a) + Df(a)\ (x-a) + \frac{1}{2}\ (x-a)^t\ D^2 f(a)\ (x-a).
$$

Here, x, a, and $x - a$ are 3×1 column vectors, i.e.,

$$
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad a = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad x - a = \begin{pmatrix} x_1 - 1 \\ x_2 + 1 \\ x_3 \end{pmatrix}.
$$

Solution: (i) Recall that

$$\begin{aligned} Df(x) &= \left(\frac{\partial f}{\partial x_1}(x) \;\; \frac{\partial f}{\partial x_2}(x) \;\; \frac{\partial f}{\partial x_n}(x) \right) \\ &= (4x_1 - x_2, \quad -x_1 + 3x_3, \quad 3x_2 - 2x_3)\,; \\ D^2 f(x) &= \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_3 \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \frac{\partial^2 f}{\partial x_3 \partial x_2}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_3}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_3}(x) & \frac{\partial^2 f}{\partial x_3^2}(x) \end{pmatrix} = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & -2 \end{pmatrix}. \end{aligned}$$

Then,

$$f(a) = f(1,-1,0) = 3 \tag{1.21}$$

$$Df(a) = Df(1,-1,0) = (5, \; -1, \; -3); \tag{1.22}$$

$$D^2 f(a) = D^2 f(1,-1,0) = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & -2 \end{pmatrix}. \tag{1.23}$$

(ii) We substitute the values from (1.21), (1.22) and (1.23) for $f(a)$, $Df(a)$ and $D^2 f(a)$, respectively, in the expression $f(a) + Df(a)\,(x-a) + \frac{1}{2}\,(x-a)^t\, D^2 f(a)\,(x-a)$ and obtain that

$$\begin{aligned} & f(a) + Df(a)\,(x-a) + \frac{1}{2}\,(x-a)^t\, D^2 f(a)\,(x-a) \\ = \;& 3 \;+\; (5,-1,-3) \begin{pmatrix} x_1 - 1 \\ x_2 + 1 \\ x_3 \end{pmatrix} \\ & + \frac{1}{2}\,(x_1 - 1, x_2 + 1, x_3) \begin{pmatrix} 4 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 + 1 \\ x_3 \end{pmatrix} \\ = \;& 3 \;+\; (5x_1 - x_2 - 3x_3 - 6) \\ & + \left(2x_1^2 - 5x_1 - x_1 x_2 + x_2 + 3x_2 x_3 + 3x_3 - x_3^2 + 3\right) \\ = \;& 2x_1^2 - x_1 x_2 + 3x_2 x_3 - x_3^2 \\ = \;& f(x). \quad \square \end{aligned}$$

Problem 16: Let

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad \text{for } \; t > 0, \; x \in \mathbb{R}.$$

Compute $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, and show that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Solution: By direct computation and using the Product Rule, we find that

$$\frac{\partial u}{\partial t} = -\frac{1}{2}t^{-3/2}\,\frac{1}{\sqrt{4\pi}}\,e^{-\frac{x^2}{4t}} + \frac{1}{\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}}\left(-\frac{x^2}{4}\cdot\left(-\frac{1}{t^2}\right)\right)$$

$$= -\frac{1}{2t\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2}\cdot\frac{1}{\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}}; \qquad (1.24)$$

$$\frac{\partial u}{\partial x} = -\frac{x}{2t}\cdot\frac{1}{\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}};$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2}\cdot\frac{1}{\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}} \qquad (1.25)$$

From (1.24) and (1.25), we conclude that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}. \quad \square$$

Problem 17: Show that the values of a plain vanilla put option and of a plain vanilla call option with the same maturity and strike, and on the same underlying asset, are equal if and only if the strike is equal to the forward price.

Solution: Recall that the forward price is $F = Se^{-(r-q)T}$.

From the Put–Call parity, we know that

$$C - P = Se^{-qT} - Ke^{-rT}. \qquad (1.26)$$

If a call and a put with the same strike K have the same value, i.e., if $C = P$ in (1.26), then $Se^{-qT} = Ke^{-rT}$. Thus,

$$K = Se^{(r-q)T},$$

i.e., the strike of the options is equal to the forward price. $\square$

Problem 18: Consider a portfolio with the following positions:
- long one call option with strike $K_1 = 30$;
- short two call options with strike $K_2 = 35$;
- long one call option with strike $K_3 = 40$.

All options are on the same underlying asset and have maturity T. Draw the payoff diagram at maturity of the portfolio, i.e., plot the value of the portfolio $V(T)$ at maturity as a function of $S(T)$, the price of the underlying asset at time T.

Solution: A butterfly spread is an options portfolio made of a long position in one call option with strike K_1, a long position in a call option with strike

K_3, and a short position in two calls with strike equal to the average of the strikes K_1 and K_3, i.e., with strike $K_2 = \frac{K_1+K_3}{2}$; all options have the same maturity and have the same underlying asset.

The payoff at maturity of a butterfly spread is always nonnegative, and it is positive if the price of the underlying asset at maturity is between the strikes K_1 and K_3, i.e., if $K_1 < S(T) < K_3$.

For our problem, the values of the options at maturity are, respectively,

$$\begin{aligned} C_1(T) &= \max(S(T) - K_1, 0) = \max(S(T) - 30, 0); \\ C_2(T) &= \max(S(T) - K_2, 0) = \max(S(T) - 35, 0); \\ C_3(T) &= \max(S(T) - K_3, 0) = \max(S(T) - 40, 0), \end{aligned}$$

and the value of the portfolio at maturity is

$$V(T) = C_1(T) - 2C_2(T) + C_3(T).$$

Depending on the values of the spot $S(T)$ of the underlying asset at maturity, the value $V(T)$ of the portfolio at time T is given below:

	$S(T) < 30$	$30 < S(T) < 35$	$35 < S(T) < 40$	$40 < S(T)$
$C_1(T)$	0	$S(T) - 30$	$S(T) - 30$	$S(T) - 30$
$C_2(T)$	0	0	$S(T) - 35$	$S(T) - 35$
$C_3(T)$	0	0	0	$S(T) - 40$
$V(T)$	0	$S(T) - 30$	$40 - S(T)$	0

Problem 19: Draw the payoff diagram at maturity of a bull spread with a long position in a call with strike 30 and short a call with strike 35, and of a bear spread with long a put of strike 20 and short a put of strike 15.

Solution: The payoff of the bull spread at maturity T is

$$V_1(T) = \max(S(T) - 30, 0) - \max(S(T) - 35, 0).$$

Depending on the value of the spot price $S(T)$, the value of the bull spread at maturity T is

	$S(T) < 30$	$30 < S(T) < 35$	$35 < S(T)$
$V_1(T)$	0	$S(T) - 30$	5

The value of the bear spread at maturity T is

$$V_2(T) = \max(20 - S(T), 0) - \max(15 - S(T), 0),$$

which can be written in terms of the value of $S(T)$ as

	$S(T) < 15$	$15 < S(T) < 20$	$20 < S(T)$
$V_2(T)$	5	$20 - S(T)$	0

A trader takes a long position in a bull spread if the underlying asset is expected to appreciate in value, and takes a long position in a bear spread if the value of the underlying asset is expected to depreciate. □

Problem 20: The prices of three call options with strikes 45, 50, and 55, on the same underlying asset and with the same maturity, are \$4, \$6, and \$9, respectively. Create a butterfly spread by going long a 45–call and a 55–call, and shorting two 50–calls. What are the payoff and the P&L at maturity of the butterfly spread? When would the butterfly spread be profitable? Assume, for simplicity, that interest rates are zero.

Solution: The payoff $V(T)$ of the butterfly spread at maturity is

$$V(T) = \begin{cases} 0, & \text{if } S(T) \leq 45; \\ S(T) - 45, & \text{if } 45 < S(T) \leq 50; \\ 55 - S(T), & \text{if } 50 < S(T) < 55; \\ 0, & \text{if } 55 \leq S(T). \end{cases}$$

The cost to set up the butterfly spread is

$$\$4 - \$12 + \$9 = \$1.$$

The P&L at maturity is equal to the payoff $V(T)$ minus the future value at time T of \$1, the setup cost. Since interest rates are zero, the future value of \$1 is \$1, and we conclude that

$$P\&L(T) = \begin{cases} -1, & \text{if } S(T) \leq 45; \\ S(T) - 46, & \text{if } 45 < S(T) \leq 50; \\ 54 - S(T), & \text{if } 50 < S(T) < 55; \\ -1, & \text{if } 55 \leq S(T). \end{cases}$$

The butterfly spread will be profitable if $46 < S(T) < 54$, i.e., if the spot price at maturity of the underlying asset will be between \$46 and \$54.

If $r \neq 0$, it follows similarly that the butterfly spread is profitable if

$$45 + e^{rT} < S(T) < 55 - e^{rT}. \quad \Box$$

Problem 21: Which of the following two portfolios would you rather hold:

- Portfolio 1: Long one call option with strike $K = X - 5$ and long one call option with strike $K = X + 5$;

- Portfolio 2: Long two call options with strike $K = X$?
  (All options are on the same asset and have the same maturity.)

Solution: Note that being long Portfolio 1 and short Portfolio 2 is equivalent to being long a butterfly spread, and therefore will always have positive (or rather nonnegative) payoff at maturity. Therefore, if you are to *assume* a position in either one of the portfolios (not to purchase the portfolios), you are better off owning Portfolio 1, since its payoff at maturity will always be at least as big as the payoff of Portfolio 2.

To see this rigorously, denote by $V_1(t)$ and $V_2(t)$ the value of Portfolio 1 and of Portfolio 2, respectively. Let $V(t) = V_1(t) - V_2(t)$ be the value of a long position in Portfolio 1 and a short position in Portfolio 2.

The value $V(T)$ of the portfolio at the maturity T of the options is given by

$$\begin{aligned} V(T) &= V_1(T) - V_2(T) \\ &= \max(S(T) - (X-5), 0) + \max(S(T) - (X+5), 0) \\ &\quad - 2\max(S(T) - X, 0). \end{aligned}$$

It is easy to see that $V(T) \geq 0$ for any value of $S(T)$, since $V(T)$ in terms of $S(T)$ is given in the table below:

	$V(T)$
$S(T) < X - 5$	0
$X - 5 < S(T) < X$	$S(T) - (X - 5)$
$X < S(T) < X + 5$	$(X + 5) - S(T)$
$X + 5 < S(T)$	0

Problem 22: A stock with spot price \$42 pays dividends continuously at a rate of 3%. The four months put and call options with strike 40 on this asset are trading at \$2 and \$4, respectively. The risk-free rate is constant and equal to 5% for all times. Show that the Put-Call parity is not satisfied and explain how would you take advantage of this arbitrage opportunity.

Solution: The following values are given: $S = 42$; $K = 40$; $T = 1/3$; $r = 0.05$; $q = 0.03$; $P = 2$; $C = 4$.

The Put–Call parity is not satisfied, since

$$P + Se^{-qT} - C \;=\; 39.5821 \;>\; 39.3389 \;=\; Ke^{-rT}. \qquad (1.27)$$

Therefore, a riskless profit can be obtained by "buying low and selling high", i.e., by selling the portfolio on the left hand side of (1.27) and buying the portfolio on the right hand side of (1.27) (which is cash only). The riskless profit at maturity will be the future value at time T of the mispricing from the Put–Call parity, i.e.,

$$(39.5821 - 39.3389)e^{rT} \;=\; 0.2473. \tag{1.28}$$

To show this, start with no money and sell one put option, short e^{-qT} shares, and buy one call option. This will generate the following cash amount:

$$P + Se^{-qT} - C \;=\; 39.5821,$$

since shorting the shares means that e^{-qT} shares are borrowed and sold on the market for cash. (The short will be closed at maturity T by buying shares on the market and returning them to the borrower; see below for more details.)

At time 0, the portfolio consists of the following positions:

- short one put option with strike K and maturity T;
- short e^{-qT} shares;
- long one call option with strike K and maturity T;
- cash: +\$39.5821.

The initial value of the portfolio is zero, since no money were invested:

$$V(0) \;=\; -P(0) - S(0)e^{-qT} + C(0) + 39.5821 \;=\; 0.$$

Note that by shorting the shares you are responsible for paying the accrued dividends. Assume that the dividend payments are financed by shorting more shares of the underlying asset and using the cash proceeds to make the dividend payments. Then, the short position in e^{-qT} shares at time 0 will become a short position in one share[3] at time T.

The value of the portfolio at maturity is

$$V(T) \;=\; -P(T) - S(T) + C(T) + 39.5821e^{rT}.$$

As shown when proving the Put–Call parity,

$$P(T)+S(T)-C(T) \;=\; \max(K-S(T),0)+S(T)-\max(S(T)-K,0) \;=\; K,$$

regardless of the value $S(T)$ of the underlying asset at maturity.

Therefore,

$$\begin{aligned} V(T) &= -(P(T) + S(T) - C(T)) + 39.5821e^{rT} \\ &= -K + 39.5821e^{rT} \;=\; -40 + 40.2473 \;=\; 0.2473. \end{aligned}$$

This value represents the risk–free profit made by exploiting the discrepancy from the Put–Call parity, and is the same as the future value at time T of the mispricing from the Put–Call parity; cf. (1.28). □

[3]This is similar to converting a long position in e^{-qT} shares at time 0 into a long position in one share at time T, through continuous purchases of (fractions of) shares using the dividend payments, which is a more intuitive process.

Problem 23: The bid and ask prices for a six months European call option with strike 40 on a non–dividend–paying stock with spot price 42 are \$5 and \$5.5, respectively. The bid and ask prices for a six months European put option with strike 40 on the same underlying asset are \$2.75 and \$3.25, respectively. Assume that the risk free rate is equal to 0. Is there an arbitrage opportunity present?

Solution: For $r = 0$, the Put–Call parity becomes $P + S - C = K$, which in this case can be written as $C - P = 2$.

Thus, an arbitrage occurs if $C - P$ can be "bought" for less than \$2 (i.e., if a call option is bought and a put option is sold for less than \$2), or if $C - P$ can be "sold" for more than \$2 (i.e., if a call option can be sold and a put option can be bought for more than \$2).

From the bid and ask prices, we find that the call can be bought for \$5.5 and the put can be sold for \$2.75. Then, $C - P$ can be "bought" for \$5.5-\$2.75=\$2.75, which is more than \$2. Therefore, no risk-free profit can be achieved this way.

Also, a call can be sold for \$5 and a put can be bought for \$3.25. Therefore, $C - P$ can be "sold" for \$5-\$3.25=\$1.75, which is less than \$2. Again, no risk-free profit can be achieved. □

Problem 24: Denote by C_{bid} and C_{ask}, and by P_{bid} and P_{ask}, respectively, the bid and ask prices for a plain vanilla European call and for a plain vanilla European put option, both with the same strike K and maturity T, and on the same underlying asset with spot price S and paying dividends continuously at rate q. Assume that the risk–free interest rates are constant equal to r.

Find necessary and sufficient no–arbitrage conditions for C_{bid}, C_{ask}, P_{bid}, and P_{ask}.

Solution: Recall the Put–Call parity

$$C - P \;=\; Se^{-qT} - Ke^{-rT},$$

where the right hand represents the value of a forward contract on the underlying asset with strike K.

An arbitrage would exist in one of the following two instances:

• if the purchase price of a long call short put portfolio, i.e., $C_{ask} - P_{bid}$ were less than the value $Se^{-qT} - Ke^{-rT}$ of the forward contract, i.e., if

$$C_{ask} - P_{bid} \;<\; Se^{-qT} - Ke^{-rT};$$

• if the selling price of a long call short put portfolio, i.e., $C_{bid} - P_{ask}$ were greater than the value $Se^{-qT} - Ke^{-rT}$ of the forward contract, i.e., if

$$C_{bid} - P_{ask} \;>\; Se^{-qT} - Ke^{-rT}.$$

We conclude that there is no–arbitrage directly following from the Put–Call parity if and only if

$$C_{bid} - P_{ask} \;\le\; Se^{-qT} - Ke^{-rT} \;\le\; C_{ask} - P_{bid}. \tag{1.29}$$

Note that the no–arbitrage condition (1.29) can also be written as

$$C_{bid} - P_{ask} \;\le\; F \;\le\; C_{ask} - P_{bid},$$

where $F = Se^{-qT} - Ke^{-rT}$ is the value of a forward contract with delivery price K and maturity T on the same underlying asset. □

Problem 25: You expect that an asset with spot price \$35 will trade in the \$40–\$45 range in one year. One year at–the–money calls on the asset can be bought for \$4. To act on the expected stock price appreciation, you decide to either buy the asset, or to buy ATM calls. Which strategy is better, depending on where the asset price will be in a year?

Solution: For every \$1000 invested, the payoff in one year of the first strategy, i.e., of buying the asset, is

$$V_1(T) \;=\; \frac{1000}{35} S(T),$$

where $S(T)$ is the spot price of the asset in one year.

For every \$1000 invested, the payoff in one year of the second strategy, i.e., of investing everything in buying call options, is

$$\begin{aligned} V_2(T) &= \frac{1000}{4}\max(S(T) - 35, 0) \\ &= \begin{cases} \frac{1000}{4}(S(T) - 35), & \text{if } S(T) \ge 35; \\ 0, & \text{if } S(T) < 35. \end{cases} \end{aligned}$$

If $S(T)$ is less than \$35, the calls expire worthless and the speculative strategy of investing everything in call options will lose all the money invested in it, while the first strategy of buying the asset will not lose all its value. However, investing everything in the call options is very profitable if the asset appreciates in value, i.e., is $S(T)$ is significantly larger than \$35. The breakeven point of the two strategies, i.e., the spot price at maturity of the underlying asset where both strategies have the same payoff is \$39.5161, since

$$\frac{1000}{35} S(T) = \frac{1000}{4}(S(T) - 35) \Longleftrightarrow S(T) = 39.5161.$$

If the price of the asset will, indeed, be in the \$40–\$45 range in one year, then buying the call options is the more profitable strategy. □

Problem 26: Create a portfolio with the following payoff at time T:

$$V(T) = \begin{cases} 2S(T), & \text{if } S(T) < 20; \\ 60 - S(T), & \text{if } 20 \leq S(T) < 40; \\ S(T) - 20, & \text{if } 40 \leq S(T), \end{cases} \qquad (1.30)$$

where $S(T)$ is the spot price at time T of a given asset. Use plain vanilla options with maturity T as well as cash positions and positions in the asset itself. Assume, for simplicity, that the asset does not pay dividends and that interest rates are zero.

Solution: Using plain vanilla options, cash, and the underlying asset the payoff $V(T)$ can be replicated in different ways.

One way is to use the underlying asset, calls with strike 20, and calls with strike 40.

First of all, a portfolio with a long position in two units of the underlying asset has value $2S(T)$ at maturity, when $S(T) < 20$.

To replicate the portfolio payoff $60 - S(T)$ when $20 \leq S(T) < 40$, note that

$$60 - S(T) = 2S(T) + 60 - 3S(T) = 2S(T) - 3(S(T) - 20).$$

This is equivalent to a long position in two units of the underlying asset and a short position in three calls with strike 20.

To replicate the portfolio payoff $S(T) - 20$ when $40 \leq S(T)$, note that

$$S(T) - 20 = 60 - S(T) + 2S(T) - 80 = 2S(T) - 3(S(T) - 20) + 2(S(T) - 40).$$

This is equivalent to a long position in two units of the underlying asset, a short position in three calls with strike 20, and a long position in two calls with strike 40.

Summarizing, the replicating portfolio is made of

- long two units of the asset;
- short 3 call options with strike $K = 20$ on the asset;
- long 2 call options with strike $K = 40$ on the asset.

We check that the payoff of this portfolio at maturity, i.e.,

$$V_1(T) = 2S(T) - 3\max(S(T) - 20, 0) + 2\max(S(T) - 40, 0) \qquad (1.31)$$

is the same as the payoff from (1.30):

	$V_1(T)$
$S(T) < 20$	$2S(T)$
$20 \leq S(T) < 40$	$2S(T) - 3(S(T) - 20) \; = \; 60 - S(T)$
$40 \leq S(T)$	$60 - S(T) + 2(S(T) - 40) \; = \; S(T) - 20$

As a consequence of the Put–Call parity, it follows that the payoff $V(T)$ from (1.30) can also be synthesized using put options. If the asset does not pay dividends and if interest rates are zero, then, from the Put–Call parity, it follows that

$$C \; = \; P + S - K.$$

Denote by C_{20} and P_{20}, and by C_{40} and P_{40}, the values of the call and put options with strikes 20 and 40, respectively.

Then, the replicating portfolio with payoff at maturity given by (1.31) can be written as

$$V \; = \; 2S - 3C_{20} + 2C_{40}. \tag{1.32}$$

To synthesize a short position in three calls with strike 20, note that

$$-3C_{20} \; = \; -3P_{20} - 3S + 60, \tag{1.33}$$

which is equivalent to taking a short position in three units of the underlying asset, taking a short position in three put options with strike 20, and being a long \$60.

Similarly, to synthesize a long position in two calls with strike 40, note that

$$2C_{40} \; = \; 2P_{40} + 2S - 80, \tag{1.34}$$

which is equivalent to a borrowing \$80, taking a long position in two units of the underlying asset, and taking a long position in two put options with strike 40.

Using (1.33) and (1.34), we obtain that the payoff at maturity given by (1.31) can be replicated using the following portfolio consisting of put options, cash, and the underlying asset:

$$\begin{aligned} V \; &= \; 2S - 3C_{20} + 2C_{40} \\ &= \; 2S - 3P_{20} - 3S + 60 + 2P_{40} + 2S - 80 \\ &= \; S - 3P_{20} + 2P_{40} - 20. \end{aligned} \tag{1.35}$$

The positions of the replicating portfolio (1.35) can be summarized as follows:

- long one unit of the asset;
- short \$20 cash;
- short 3 put options with strike $K = 20$ on the asset;
- long 2 put options with strike $K = 40$ on the asset.

We check that the payoff of this portfolio at maturity, i.e.,

$$V_2(T) \;=\; S(T) - 20 - 3\max(20 - S(T), 0) + 2\max(40 - S(T), 0)$$

is the same as the payoff from (1.30):

	$V_1(T)$
$S(T) \leq 20$	$S(T) - 20 - 3(20 - S(T)) + 2(40 - S(T)) \;=\; 2S(T)$
$20 < S(T) \leq 40$	$S(T) - 20 + 2(40 - S(T)) \;=\; 60 - S(T)$
$40 < S(T)$	$S(T) - 20$

If the asset pays dividends continuously at rate q and if interest rates are constant and equal to r, in order to obtain the same payoffs at maturity, the asset positions in the two portfolios must be adjusted as follows:

The first replicating portfolio will be made of the following positions:
- long $2e^{-qT}$ units of the asset;
- short 3 call options with strike $K = 20$ on the asset;
- long 2 call options with strike $K = 40$ on the asset.

The second replicating portfolio will be made of the following positions:
- long e^{-qT} units of the asset;
- short \$$20e^{-rT}$ cash;
- short 3 put options with strike $K = 20$ on the asset;
- long 2 put options with strike $K = 40$ on the asset.

Note that any piecewise linear payoff of a single asset can be synthesized, in theory, by using plain vanilla options, cash and asset positions. □

Problem 27: A derivative security pays a cash amount c if the spot price of the underlying asset at maturity is between K_1 and K_2, where $0 < K_1 < K_2$, and expires worthless otherwise. How do you synthesize this derivative security (i.e., how do you recreate its payoff almost exactly) using plain vanilla call options?

Solution: The payoff of the derivative security is

$$V(T) \;=\; \begin{cases} 0, & \text{if } S(T) \leq K_1; \\ c, & \text{if } K_1 < S(T) < K_2; \\ 0, & \text{if } K_2 \leq S(T). \end{cases}$$

Since $V(T)$ is discontinuous, it cannot be replicated exactly using call options, whose payoffs are continuous.

We approximate the payoff $V(T)$ of the derivative security by the following payoff

$$V_\epsilon(T) = \begin{cases} 0, & \text{if } S(T) < K_1 - \epsilon; \\ \frac{c}{\epsilon}(S(T) - (K_1 - \epsilon)), & \text{if } K_1 - \epsilon \le S(T) \le K_1; \\ c, & \text{if } K_1 < S(T) < K_2; \\ c - \frac{c}{\epsilon}(S(T) - K_2), & \text{if } K_2 \le S(T) \le K_2 + \epsilon; \\ 0, & \text{if } K_2 + \epsilon < S(T). \end{cases} \quad (1.36)$$

Note that $V(T) = V_\epsilon(T)$ unless the value $S(T)$ of the underlying asset at maturity is either between $K_1 - \epsilon$ and K_1, or between K_2 and $K_2 + \epsilon$.

The payoff $V_\epsilon(T)$ can be realized by going long c/ϵ bull spreads with strikes $K_1 - \epsilon$ and K_1, and shorting c/ϵ bull spreads with strikes K_2 and $K_2 + \epsilon$.

The payoff $V(T)$ of the given derivative security can be synthesized by taking the following positions:

- long c/ϵ calls with strike $K_1 - \epsilon$;
- short c/ϵ calls with strike K_1;
- short c/ϵ calls with strike K_2;
- long c/ϵ calls with strike $K_2 + \epsilon$.

It is easy to see that the payoff $V_\epsilon(T)$ is the same as in (1.36):

	$V_\epsilon(T)$
$S(T) < K_1 - \epsilon$	0
$K_1 - \epsilon \le S(T) < K_1$	$\frac{c}{\epsilon}(S(T) - (K_1 - \epsilon))$
$K_1 \le S(T) < K_2$	$\frac{c}{\epsilon}(S(T) - (K_1 - \epsilon)) - \frac{c}{\epsilon}(S(T) - K_1)) = c$
$K_2 \le S(T) < K_2 + \epsilon$	$c - \frac{c}{\epsilon}(S(T) - K_2)$
$K_2 + \epsilon < S(T)$	$c - \frac{c}{\epsilon}(S(T) - K_2) + \frac{c}{\epsilon}(S(T) - (K_2 + \epsilon)) = 0$

Problem 28: Call options with strikes 100, 120, and 130 on the same underlying asset and with the same maturity are trading for 8, 5, and 3, respectively (there is no bid–ask spread). Is there an arbitrage opportunity present? If yes, how can you make a riskless profit?

Note: A solution to this problem based on the convexity of the payoff of call and put options is discussed in section 1.3 at the end of this chapter.

Solution: For an arbitrage opportunity to be present, there must be a portfolio made of the three options with nonnegative payoff at maturity and with a negative cost of setting up.

Let $K_1 = 100 < K_2 = 120 < K_3 = 130$ be the strikes of the options. Denote by x_1, x_2, x_3 the options positions (which can be either negative or positive) at time 0. Then, at time 0, the portfolio is worth

$$V(0) \;=\; x_1 C_1(0) \;+\; x_2 C_2(0) \;+\; x_3 C_3(0)$$

At maturity T, the value of the portfolio will be

$$\begin{aligned} V(T) &= x_1 C_1(T) \;+\; x_2 C_2(T) \;+\; x_3 C_3(T) \\ &= x_1 \max(S(T) - K_1, 0) \;+\; x_2 \max(S(T) - K_2, 0) \\ &\quad + x_3 \max(S(T) - K_3, 0), \end{aligned}$$

respectively.

Depending on the value $S(T)$ of the underlying asset at maturity, the value $V(T)$ of the portfolio is as follows:

	$V(T)$
$S(T) < K_1$	0
$K_1 < S(T) < K_2$	$x_1 S(T) - x_1 K_1$
$K_2 < S(T) < K_3$	$(x_1 + x_2)S(T) - x_1 K_1 - x_2 K_2$
$K_3 < S(T)$	$(x_1 + x_2 + x_3)S(T) - x_1 K_1 - x_2 K_2 - x_3 K_3$

Note that $V(T)$ is nonnegative when $S(T) \leq K_2$ only if a long position is taken in the option with strike K_1, i.e., if $x_1 \geq 0$. The payoff $V(T)$ decreases when $K_2 < S(T) < K_3$, accounting for the short position in the two call options with strike K_2, and then increases when $S(T) \geq K_3$.

We conclude that $V(T) \geq 0$ for any value of S(T) if and only if $x_1 \geq 0$, if the value of the portfolio when $S(T) = K_3$ is nonnegative, i.e., if $(x_1 + x_2)K_3 - x_1 K_1 - x_2 K_2 \geq 0$, and if $x_1 + x_2 + x_3 \geq 0$.

Thus, an arbitrage exists if and only if the values $C_1(0)$, $C_2(0)$, $C_3(0)$ are such that we can find x_1, x_2, and x_3 with the following properties:

$$\begin{aligned} x_1 C_1(0) + x_2 C_2(0) + x_3 C_3(0) &< 0; \\ x_1 &\geq 0; \\ (x_1 + x_2)K_3 - x_1 K_1 - x_2 K_2 &\geq 0; \\ x_1 + x_2 + x_3 &\geq 0. \end{aligned}$$

For $C_1(0) = 8$, $C_2(0) = 5$, $C_3(0) = 3$ and $K_1 = 100$, $K_2 = 120$, $K_3 = 130$, the problem becomes finding $x_1 \geq 0$, and x_2 and x_3 such that

$$8x_1 + 5x_2 + 3x_3 \;<\; 0; \qquad (1.37)$$

$$30x_1 + 10x_2 \;\geq\; 0; \qquad (1.38)$$

$$x_1 + x_2 + x_3 \;\geq\; 0. \qquad (1.39)$$

(For these option prices, arbitrage will be possible since the middle option is overpriced relative to the other two options.)

The easiest way to find values of x_1, x_2, and x_3 satisfying the constraints above is to note that arbitrage can occur for a portfolio with long positions in the options with lowest and highest strikes, and with a short position in the option with middle strike (note the similarity to butterfly spreads). Then, choosing $x_3 = -x_1 - x_2$ would be optimal; cf. (1.39). The constraints (1.37) and (1.38) become

$$\begin{aligned} 5x_1 + 2x_2 &< 0; \\ 3x_1 + x_2 &\geq 0. \end{aligned}$$

These constraints are satisfied, e.g., for $x_1 = 1$ and $x_2 = -3$, which corresponds to $x_3 = 2$.

Buying one option with strike 100, selling three options with strike 120, and buying two options with strike 130 will generate a positive cash flow of \$1, and will result in a portfolio that will not lose money, regardless of the value of the underlying asset at the maturity of the options. □

Problem 29: Call options on the same underlying asset and with the same maturity, with strikes $K_1 < K_2 < K_3$, are trading for C_1, C_2 and C_3, respectively (no Bid–Ask spread), with $C_1 > C_2 > C_3$. Find necessary and sufficient conditions on the prices C_1, C_2 and C_3 such that no–arbitrage exists corresponding to a portfolio made of positions in the three options.

Solution: An arbitrage exists if and only if a no–cost portfolio can be set up with non–negative payoff at maturity regardless of the price of the underlying asset at maturity, and such that the probability of a strictly positive payoff is greater than 0.

Consider a portfolio made of positions in the three options with value 0 at inception, and let $x_i > 0$ be the size of the portfolio position in the option with strike K_i, for $i = 1:3$. Let $S = S(T)$ be the value of the underlying asset at maturity. For no–arbitrage to occur, there are three possibilities:

<u>Portfolio 1:</u> Long the K_1–option, short the K_2–option, long the K_3–option.

Arbitrage exists if we can find $x_i > 0$, $i = 1:3$, such that

$$x_1C_1 - x_2C_2 + x_3C_3 \;=\; 0; \tag{1.40}$$

$$x_1(S - K_1) - x_2(S - K_2) + x_3(S - K_3) \;\geq\; 0, \quad \forall\, S \geq 0. \tag{1.41}$$

We note that (1.41) holds if and only if the following two conditions are satisfied:

$$x_1 - x_2 + x_3 \;\geq\; 0; \tag{1.42}$$

$$x_1(K_3 - K_1) - x_2(K_3 - K_2) \;\geq\; 0. \tag{1.43}$$

We solve (1.40) for x_3 and obtain

$$x_3 \;=\; x_2\frac{C_2}{C_3} \;-\; x_1\frac{C_1}{C_3}. \qquad (1.44)$$

Since we assumed that $x_3 > 0$, the following condition must also be satisfied:

$$\frac{x_2}{x_1} \;>\; \frac{C_1}{C_2}. \qquad (1.45)$$

Recall that $C_1 > C_2 > C_3$. Using the value of x_3 from (1.44), it follows that (1.42) and (1.43) hold true if and only if

$$\frac{x_2}{x_1} \;\geq\; \frac{C_1 - C_3}{C_2 - C_3}; \qquad (1.46)$$

$$\frac{x_2}{x_1} \;\leq\; \frac{K_3 - K_1}{K_3 - K_2}. \qquad (1.47)$$

Also, note that if (1.46) holds true, then (1.45) is satisfied as well, since

$$\frac{C_1 - C_3}{C_2 - C_3} \;>\; \frac{C_1}{C_2}.$$

We conclude that arbitrage happens if and only if we can find $x_1 > 0$ and $x_2 > 0$ such that (1.46) and (1.47) are simultaneously satisfied. Therefore, no–arbitrage exists if and only if

$$\frac{K_3 - K_1}{K_3 - K_2} \;<\; \frac{C_1 - C_3}{C_2 - C_3}. \qquad (1.48)$$

Portfolio 2: Long the K_1–option, short the K_2–option, short the K_3–option.

Arbitrage exists if we can find $x_i > 0$, $i = 1 : 3$, such that

$$x_1 C_1 - x_2 C_2 - x_3 C_3 \;=\; 0; \qquad (1.49)$$

$$x_1(S - K_1) - x_2(S - K_2) - x_3(S - K_3) \;\geq\; 0, \;\; \forall\, S \geq 0. \qquad (1.50)$$

The inequality (1.50) holds if and only if the following two conditions are satisfied:

$$x_1 - x_2 - x_3 \;\geq\; 0; \qquad (1.51)$$

$$x_1(K_3 - K_1) - x_2(K_3 - K_2) \;\geq\; 0. \qquad (1.52)$$

However, (1.49) and (1.51) cannot be simultaneously satisfied. Since $C_1 > C_2 > C_3$, it is easy to see that

$$x_1 \;=\; x_2\frac{C_2}{C_1} - x_3\frac{C_3}{C_1} \;<\; x_2 + x_3.$$

In other words, no arbitrage can be obtained by being long the option with strike K_1 and short the options with strikes K_2 and K_3.

Portfolio 3: Long the K_1–option, long the K_2–option, short the K_3–option.

Arbitrage exists if we can find $x_i > 0$, $i = 1 : 3$, such that

$$x_1 C_1 + x_2 C_2 - x_3 C_3 \;=\; 0; \tag{1.53}$$

$$x_1(S - K_1) + x_2(S - K_2) - x_3(S - K_3) \;\geq\; 0, \quad \forall\, S \geq 0. \tag{1.54}$$

The inequality (1.54) holds if and only if

$$x_1 + x_2 - x_3 \;\geq\; 0. \tag{1.55}$$

It is easy to see that (1.53) and (1.55) cannot be simultaneously satisfied:

$$x_3 \;=\; x_1 \frac{C_1}{C_3} + x_2 \frac{C_2}{C_3} \;>\; x_1 + x_2,$$

since $C_1 > C_2 > C_3$.

Therefore, no arbitrage can be obtained by being long the options with strikes K_1 and K_2 and short the option with strike K_3.

We conclude that (1.48), i.e.,

$$\frac{K_3 - K_1}{K_3 - K_2} \;<\; \frac{C_1 - C_3}{C_2 - C_3} \tag{1.56}$$

is the only condition required for no–arbitrage. □

Problem 30: The risk free rate is 8% compounded continuously and the dividend yield of a stock index is 3%. The index is at 12,000 and the futures price of a contract deliverable in three months is 12,100. Is there an arbitrage opportunity, and how do you take advantage of it?

Solution: The arbitrage–free futures price of the futures contract is

$$12000 e^{(r-q)T} \;=\; 12000 e^{(0.08-0.03)/4} \;=\; 12150.94 \;>\; 12100.$$

Therefore, the futures contract is underpriced and should be bought while hedged statically by shorting $e^{-qT} = 0.9925$ units of index for each futures contract that is sold.

At maturity, the asset is bought for 12100 and the short is closed (the dividends paid on the short position increase the size of the short position to 1 unit of the index). The realized gain is the interest accrued on the cash resulting from the short position minus 12100, i.e.,

$$e^{0.08/4}\,(e^{-0.03/4} \cdot 12000) \;-\; 12100 \;=\; 50.94. \quad \square$$