Chapter 1

First Look: Ten Questions.

1. Put options with strikes 30 and 20 on the same underlying asset and with the same maturity are trading for \$6 and \$4, respectively. Can you find an arbitrage?

2. The number 2^{29} has 9 digits, all different. Without computing 2^{29}, find the missing digit.

3. Let W_t be a Wiener process, and let

$$X_t \;=\; \int_0^t W_\tau d\tau.$$

   What is the distribution of X_t? Is X_t a martingale?

4. Alice and Bob stand at opposite ends of a straight line segment. Bob sends 50 ants towards Alice, one after another. Alice sends 20 ants towards Bob. All ants travel along the straight line segment. Whenever two ants collide, they simply bounce back and start traveling in the opposite direction. How many

ants reach Bob and how many ants reach Alice? How many ant collisions take place?

5. Find all the values of ρ such that

$$\begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix}$$

is a correlation matrix.

6. How many independent random variables uniformly distributed on $[0,1]$ should you generate to ensure that there is at least one between 0.70 and 0.72 with probability 95%?

7. Show that the probability density function of the standard normal integrates to 1.

8. Assume the Earth is perfectly spherical and you are standing somewhere on its surface. You travel exactly 1 mile south, then 1 mile east, then 1 mile north. Surprisingly, you find yourself back at the starting point. If you are not at the North Pole, where can you possibly be?!

9. Solve the Ornstein-Uhlenbeck SDE

$$dr_t \;=\; \lambda(\theta - r_t)dt \;+\; \sigma dW_t,$$

which is used, e.g., in the Vasicek model for interest rates.

10. Write a C++ function that computes the n-th Fibonacci number.

Solutions

Question 1. Put options with strikes 30 and 20 on the same underlying asset and with the same maturity are trading for \$6 and \$4, respectively. Can you find an arbitrage?

Answer: Since the value of a put option with strike 0 is \$0, we in fact know the prices of put options with three different strikes, i.e.,

$$P(30) = 6;\ P(20) = 4;\ P(0) = 0,$$

where $P(K)$ denotes the value of a put option with strike K.

In the plane $(K, P(K))$, these option values correspond to the points $(30, 6)$, $(20, 4)$, and $(0, 0)$, which are on the line $P(K) = \frac{2}{3}K$.

This contradicts the fact that put options are strictly convex functions of strike price, and creates an arbitrage opportunity.

The arbitrage comes from the fact that the put with strike 20 is overpriced. Using a "buy low, sell high" strategy, we could buy (i.e., go long) $\frac{2}{3}$ put options with strike 30, and sell (i.e., go short) 1 put option with strike 20. To avoid fractions, we set up the following portfolio:

- long 2 puts with strike 30;
- short 3 puts with strike 20.

This portfolio is set up at no initial cost, since the cash flow generated by selling 3 puts with strike 20 and buying 2 puts with strike 30 is \$0:

$$3 \cdot \$4 - 2 \cdot \$6 \ = \ \$0.$$

At the maturity T of the options, the value of the portfolio is

$$V(T) \ = \ 2\max(30 - S(T), 0) - 3\max(20 - S(T), 0).$$

Note that $V(T)$ is nonnegative for any value $S(T)$ of the underlying asset:

If $S(T) \geq 30$, then both put options expire worthless, and $V(T) = 0$.

If $20 \leq S(T) < 30$, then

$$V(T) \;=\; 2(30 - S(T)) \;>\; 0.$$

If $0 < S(T) < 20$, then

$$\begin{aligned} V(T) &= 2(30 - S(T)) - 3(20 - S(T)) \\ &= S(T) \\ &> 0. \end{aligned}$$

In other words, we took advantage of the existing arbitrage opportunity by setting up, at no initial cost, a portfolio with nonnegative payoff at T regardless of the price $S(T)$ of the underlying asset, and with a strictly positive payoff if $0 < S(T) < 30$. $\square$

Question 2. The number 2^{29} has 9 digits, all different. Without computing 2^{29}, find the missing digit.

Answer: For any positive integer n, denote by $D(n)$ the sum of the digits of n. Recall that the difference between a number and the sum of its digits is divisible by 9, i.e.,

$$9 \mid n - D(n);$$

see the footnote below[1] for details.

[1] If the digits of n are $a_k,\ a_{k-1},\ \ldots a_1,\ a_0$ (from left to right), then

$$\begin{aligned} n &= a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \ldots a_1 \cdot 10 + a_0; \\ D(n) &= a_k + a_{k-1} + \ldots + a_1 + a_0. \end{aligned}$$

Hence,

$$n - D(n) = \sum_{i=0}^{k} a_i \cdot (10^i - 1).$$

Since $10^i - 1$ is an i-digit number with all digits equal to 9, it follows that $9 \mid 10^i - 1$, for all $i = 1 : k$, and therefore $9 \mid n - D(n)$.

Thus, for $n = 2^{29}$, it follows that

$$9 \mid 2^{29} - D\left(2^{29}\right). \tag{1.1}$$

We are given that 2^{29} has 9 digits, and that all 9 digits are different. Denote by x the missing digit. Then,

$$D\left(2^{29}\right) = \left(\sum_{j=0}^{9} j\right) - x = 45 - x. \tag{1.2}$$

From (1.1) and (1.2), it follows that

$$9 \mid 2^{29} - (45 - x). \tag{1.3}$$

Note that

$$\begin{aligned} 2^{29} &= 2^5 \cdot (2^6)^4 \ = \ 2^5 \cdot 64^4 \\ &= 2^5 \cdot (63+1)^4 \\ &= 2^5 \cdot (63 \cdot k + 1) \\ &= 2^5 \cdot 63 \cdot k + 2^5, \end{aligned} \tag{1.4}$$

where k is a positive integer.[2]

From (1.4), we find that

$$2^{29} - 2^5 \ = \ 63 \cdot 2^5 \cdot k,$$

and therefore

$$9 \mid 2^{29} - 2^5. \tag{1.5}$$

From (1.3) and (1.5), it follows that

$$\begin{aligned} 9 \quad \mid \quad & (2^{29} - 2^5) \ - \ (2^{29} - (45 - x)) \\ & = (45 - x) - 2^5 \\ & = 13 - x. \end{aligned}$$

[2]It is easy to see that

$$\begin{aligned} (63+1)^4 &= 63^4 + 4 \cdot 63^3 + 6 \cdot 63^2 + 4 \cdot 63 + 1 \\ &= 63 \cdot (63^3 + 4 \cdot 63^2 + 6 \cdot 63 + 4) + 1 \\ &= 63 \cdot k + 1, \end{aligned}$$

where $k = 63^3 + 4 \cdot 63^2 + 6 \cdot 63 + 4$.

Since $9 \mid 13-x$ and x is a digit, we conclude that $x = 4$. In other words, we identified that x, the missing digit from 2^{29}, must be 4.

Indeed, $2^{29} = 536\,870\,912$, i.e., 2^{29} has 9 digits, all different, and 4 is not a digit of 2^{29}. $\Box$

Question 3. Let W_t be a Wiener process, and let

$$X_t = \int_0^t W_\tau d\tau. \tag{1.6}$$

What is the distribution of X_t? Is X_t a martingale?

Answer: Note that we can rewrite (1.6) in differential form as

$$dX_t = W_t dt = W_t dt + 0\,dW_t.$$

Then, X_t is a diffusion process with only drift part W_t, and therefore X_t is not a martingale.

We use integration by parts to find the distribution of X_t; a different solution can be found in Section 3.6.

By applying integration by parts, we obtain that

$$\begin{aligned} X_t &= \int_0^t W_\tau d\tau \\ &= tW_t - \int_0^t \tau dW_\tau \\ &= t\int_0^t dW_\tau - \int_0^t \tau dW_\tau \\ &= \int_0^t (t-\tau) dW_\tau. \end{aligned}$$

Recall that, if $f(t)$ is a deterministic square integrable function, then the stochastic integral $\int_0^t f(\tau)dW_\tau$ is a normal random variable of mean 0 and variance $\int_0^t |f(\tau)|^2 d\tau$, i.e.,

$$\int_0^t f(\tau)dW_\tau \sim N\left(0, \int_0^t |f(\tau)|^2 d\tau\right).$$

Thus,

$$\begin{aligned} X_t &= \int_0^t (t-\tau) dW_\tau \\ &\sim N\left(0, \int_0^t (t-\tau)^2 \, d\tau\right) \\ &= N\left(0, \frac{t^3}{3}\right). \end{aligned}$$

We conclude that X_t is a normal random variable of mean 0 and variance $\frac{t^3}{3}$. $\square$

Question 4. Alice and Bob stand at opposite ends of a straight line segment. Bob sends 50 ants towards Alice, one after another. Alice sends 20 ants towards Bob. All ants travel along the straight line segment. Whenever two ants collide, they simply bounce back and start traveling in the opposite direction. How many ants reach Bob and how many ants reach Alice? How many ant collisions take place?

Answer: Imagine that when two ants meet, they switch identities. Hence, even after a collision, two ants are traveling in two opposite directions. It follows that 20 ants reach Bob, while 50 ants reach Alice.

To calculate the number of ant collisions, imagine that each ant carries a message. In other words, Bob sends 50 messages to Alice, one message per ant. Similarly, Alice sends 20 messages to Bob, one message per ant. Furthermore, imagine that the two ants swap messages when they collide. Then a message always makes forward progress. Each of Alice's messages goes through 50 ant collisions. Each of Bob's messages goes through 20 ant collisions. The total number of collisions is 50 times 20, which is 1000 collisions. $\square$

Question 5. Find all the values of ρ such that

$$\begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix}$$

is a correlation matrix.

Answer: A symmetric matrix with diagonal entries equal to 1 is a correlation matrix if and only if the matrix is symmetric positive semidefinite. Thus, we need to find all the values of ρ such that the matrix

$$\Omega = \begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix} \tag{1.7}$$

is symmetric positive semidefinite.

We give a short solution using Sylvester's criterion. Two more solutions, one using the Cholesky decomposition, and another one based on the definition of symmetric positive semidefinite matrices will be given in Section 3.2.

Recall from Sylvester's criterion that a matrix is symmetric positive semidefinite if and only if all its principal minors are greater than or equal to 0. Also, recall that the principal minors of a matrix are the determinants of all the square matrices obtained by eliminating the same rows and columns from the matrix. In particular, the matrix Ω from (1.7) has the following principal minors:

$$\det(1) = 1; \quad \det(1) = 1; \quad \det(1) = 1;$$

$$\begin{aligned} \det\begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix} &= 0.64; \\ \det\begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix} &= 0.91; \\ \det\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} &= 1 - \rho^2; \end{aligned}$$

$$\begin{aligned} \det(\Omega) &= 1 - 0.36\rho - 0.09 - 0.36 - \rho^2 \\ &= 0.55 - 0.36\rho - \rho^2. \end{aligned}$$

Thus, it follows from Sylvester's criterion that Ω is a symmetric positive semidefinite matrix if and only if

$$\begin{aligned} 1 - \rho^2 &\geq 0; \\ 0.55 - 0.36\rho - \rho^2 &\geq 0, \end{aligned}$$

which is equivalent to $-1 \leq \rho \leq 1$ and

$$\rho^2 + 0.36\rho - 0.55 \leq 0. \tag{1.8}$$

Since the roots of the quadratic equation corresponding to (1.8) are -0.9432 and 0.5832, we conclude that the matrix Ω is symmetric positive semidefinite, and therefore a correlation matrix, if and only if

$$-0.9432 \leq \rho \leq 0.5832. \quad \square \tag{1.9}$$

Question 6. How many independent random variables uniformly distributed on $[0, 1]$ should you generate to ensure that there is at least one between 0.70 and 0.72 with probability 95%?

Answer: Denote by N the smallest number of random variables you should generate such that

$$P(\text{at least one r.v. in } [0.70, 0.72]) \geq 0.95. \tag{1.10}$$

The probability that a random variable uniformly distributed on $[0, 1]$ is not in the interval $[0.70, 0.72]$ is 0.98. Thus, the probability that none of the N independent variables are in $[0.70, 0.72]$ is 0.98^N, i.e.,

$$P(\text{no r.v. in } [0.70, 0.72]) = 0.98^N.$$

Note that

$$\begin{aligned}
& P(\text{at least one r.v. in } [0.70, 0.72]) \\
= \; & 1 - P(\text{no r.v. in } [0.70, 0.72]) \\
= \; & 1 - (0.98)^N. \qquad (1.11)
\end{aligned}$$

From (1.10) and (1.11), we find that N is the smallest integer such that

$$1 - (0.98)^N \;\geq\; 0.95,$$

which is equivalent to

$$\begin{aligned}
& (0.98)^N \leq 0.05 \\
\iff \; & N \ln(0.98) \leq \ln(0.05) \\
\iff \; & N \geq \frac{\ln(0.05)}{\ln(0.98)} \approx 148.28 \\
\iff \; & N = 149.
\end{aligned}$$

We conclude that at least 149 uniform random variables on $[0, 1]$ must be generated in order to have 95% confidence that at least one of the random variables is between 0.70 and 0.72. $\square$

Question 7. Show that the probability density function of the standard normal integrates to 1.

Answer: The probability density function of the standard normal variable is $\frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}$. We want to show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt \;=\; 1,$$

which, using the substitution $t = \sqrt{2}x$, can be written as

$$I \;=\; \int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}. \qquad (1.12)$$

We prove (1.12) by using polar coordinates. Since x is just an integrating variable, we can also write the integral I in terms of another integrating variable, denoted by y, as $I = \int_{-\infty}^{\infty} e^{-y^2} dy$.

Then,[3]

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dxdy \qquad (1.13) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy. \end{aligned}$$

We use the polar coordinates transformation $x = r\cos\theta$ and $y = r\sin\theta$, with $r \in [0,\ \infty)$ and $\theta \in [0,\ 2\pi)$, to evaluate the last integral. Since $dxdy = rd\theta dr$, we obtain that

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-\left(r^2\cos^2\theta + r^2\sin^2\theta\right)}\, d\theta dr \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-r^2}\, d\theta dr \qquad (1.14) \\ &= \int_0^{\infty} 2\pi\ r\ e^{-r^2}\, dr \\ &= 2\pi \lim_{t\to\infty} \int_0^t r\ e^{-r^2}\, dr \\ &= 2\pi \lim_{t\to\infty} \left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^t \\ &= \pi; \end{aligned}$$

note that (1.14) follows from the equality $\cos^2\theta + \sin^2\theta = 1$ for any real number θ.

[3]Note that Fubini's theorem is needed for a rigorous derivation of the equality (1.13); this technical step is rarely required by the interviewer.

Since $I > 0$, $I = \sqrt{\pi}$, which is what we wanted to prove; see (1.12). □

Question 8. Assume the Earth is perfectly spherical and you are standing somewhere on its surface. You travel exactly 1 mile south, then 1 mile east, then 1 mile north. Surprisingly, you find yourself back at the starting point. If you are not at the North Pole, where can you possibly be?!

Answer: There are infinitely many locations, aside from the North Pole, that have this property.

Somewhere near the South Pole, there is a latitude that has a circumference of one mile. In other words, if you are at this latitude and start walking east (or west), in one mile you will be back exactly where you started from. If you instead start at some point one mile north of this latitude, your journey will take you one mile south to this special latitude, then one mile east "around the globe" and finally one mile north right back to wherever you started from. Moreover, there are infinitely many points on the Earth that are one mile north of this special latitude, where you could start your journey and eventually end up exactly where you started.

We are still not finished! There are infinitely many special latitudes as well; namely, you could start at any point one mile north of the latitude that has a circumference of $1/k$ miles, where k is a positive integer. Your journey will take you one mile south to this special latitude, then one mile east looping "around the globe" k times, and finally one mile north right back to where you started from. □

Question 9. Solve the Ornstein-Uhlenbeck SDE

$$dr_t = \lambda(\theta - r_t)dt + \sigma dW_t, \tag{1.15}$$

which is used, e.g., in the Vasicek model for interest rates.

Answer: We can rewrite (1.15) as

$$dr_t + \lambda r_t dt \;=\; \lambda\theta dt + \sigma dW_t. \tag{1.16}$$

By multiplying (1.16) on both sides by the integrating factor $e^{\lambda t}$, we obtain that

$$e^{\lambda t} dr_t + \lambda e^{\lambda t} r_t dt \;=\; \lambda\theta e^{\lambda t} dt + \sigma e^{\lambda t} dW_t,$$

which is equivalent to

$$d\left(e^{\lambda t} r_t\right) \;=\; \lambda\theta e^{\lambda t} dt + \sigma e^{\lambda t} dW_t. \tag{1.17}$$

By integrating (1.17) from 0 to t, it follows that

$$\begin{aligned} e^{\lambda t} r_t - r_0 &= \lambda\theta \int_0^t e^{\lambda s} ds + \sigma \int_0^t e^{\lambda s} dW_s \\ &= \theta\left(e^{\lambda t} - 1\right) + \sigma \int_0^t e^{\lambda s} dW_s\,. \end{aligned}$$

By solving for r_t, we find that the solution to the Ornstein-Uhlenbeck SDE is

$$\begin{aligned} r_t &= e^{-\lambda t} r_0 + e^{-\lambda t}\theta\left(e^{\lambda t} - 1\right) + \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dW_s \\ &= e^{-\lambda t} r_0 + \theta\left(1 - e^{-\lambda t}\right) + \sigma \int_0^t e^{-\lambda(t-s)} dW_s. \end{aligned}$$

Note that the process r_t is mean reverting to θ, regardless of the starting point r_0. To see this, recall that the expected value of the stochastic integral $\int_0^t f(s) dW_s$ of a non-random function $f(s)$ is 0. Then,

$$E\left[\int_0^t e^{-\lambda(t-s)} dW_s\right] = 0,$$

and therefore

$$E[r_t] \;=\; e^{-\lambda t} r_0 + \theta\left(1 - e^{-\lambda t}\right).$$

Thus,

$$\lim_{t \to \infty} E[r_t] \;=\; \theta. \quad \Box$$

Question 10. Write a C++ function that computes the n-th Fibonacci number.

Answer: The Fibonacci numbers $(F_n)_{n \geq 0}$ are given by the following recurrence:

$$F_{n+2} \;=\; F_{n+1} + F_n, \quad \forall\, n \geq 0,$$

with $F_0 = 0$ and $F_1 = 1$.

```
//recursive implementation
int fib(int n) {
    if (n == 0 || n == 1) return n;
    else {
        return fib(n-1) + fib(n-2);
    }
}

//iterative implementation
int fib(int n ){
    if (n == 0 || n == 1) return n;
    int prev = 0, last = 1, temp;
    for (int i = 2; i <= n; ++i) {
        temp = last;
        last = prev + last;
        prev = temp;
    }
    return last;
}

//tail recursive implementation
int fib(int n, int last = 1, int prev = 0)
{
    if (n == 0) return prev;
```

```
    if (n == 1) return last;
        return fib(n-1, last+prev, last);
}
```