Example: As a simple application of the Linear Transformation Property, let X_1, X_2, ..., X_n be nonconstant random variables, and let Y_1, Y_2, ..., Y_n be the random variables given by $Y_i = d_i X_i$, where $d_i \neq 0$, are constants, for $i = 1 : n$. Let $\mathbf{Y} = (Y_i)_{i=1:n}$ and $\mathbf{X} = (X_i)_{i=1:n}$. Then, $\mathbf{Y} = D\mathbf{X}$, and, from the Linear Transformation Property (7.74), it follows that $\Sigma_{\mathbf{Y}} = D\Sigma_{\mathbf{X}}D^t$. Since D is a diagonal matrix, $D^t = D$, we conclude that

$$\Sigma_{\mathbf{Y}} \;=\; D\Sigma_{\mathbf{X}}D. \quad \Box \tag{7.81}$$

An extension of the Linear Transformation Property can be found below:

Theorem 7.4. *Let X_1, X_2, ..., X_n be n random variables, and let $\boldsymbol{X} = (X_i)_{i=1:n}$. Denote by $\mu_{\boldsymbol{X}}$ the mean vector of $\boldsymbol{X}$ and by $\Sigma_{\boldsymbol{X}}$ the covariance matrix of $\boldsymbol{X}$. Let Y_1, Y_2, ..., Y_m be random variables given by*

$$\boldsymbol{Y} \;=\; b + M\boldsymbol{X},$$

where $\boldsymbol{Y} = (Y_i)_{i=1:m}$, M is an $m \times n$ matrix, and b is an $m \times 1$ column vector. Denote by $\mu_{\boldsymbol{Y}}$ the mean vector of $\boldsymbol{Y}$ and by $\Sigma_{\boldsymbol{Y}}$ the covariance matrix of $\boldsymbol{Y}$.

Then,

$$\mu_{\boldsymbol{Y}} \;=\; b + M\mu_{\boldsymbol{X}} \quad \text{and} \quad \Sigma_{\boldsymbol{Y}} \;=\; M\Sigma_{\boldsymbol{X}}M^t. \tag{7.82}$$

Proof. From the linearity of expectation, it follows that

$$E[M\mathbf{X}] \;=\; ME[\mathbf{X}] \;=\; M\mu_{\mathbf{X}}; \tag{7.83}$$

see (7.126). Then, from (7.83), we find that

$$\mu_{\mathbf{Y}} \;=\; E[\mathbf{Y}] \;=\; E[b + M\mathbf{X}] \;=\; b + E[M\mathbf{X}] \;=\; b + M\mu_{\mathbf{X}}.$$

Moreover, since b is a constant vector, the covariance matrix of $\mathbf{Y} = b + M\mathbf{X}$ is the same as the covariance matrix of $M\mathbf{X}$, and therefore $\Sigma_{\mathbf{Y}} = \Sigma_{M\mathbf{X}}$. Note that $\Sigma_{M\mathbf{X}} = M\Sigma_{\mathbf{X}}M^t$; see (7.75). Thus, we conclude that

$$\Sigma_{\mathbf{Y}} \;=\; \Sigma_{M\mathbf{X}} \;=\; M\Sigma_{\mathbf{X}}M^t.$$

$\Box$

The Linear Transformation Property can be used for generating normal random variables with a given correlation matrix, which can be subsequently used for Monte Carlo simulations, see section 7.5, and for establishing that any symmetric positive semidefinite matrix is the covariance matrix (or the correlation matrix, if the main diagonal entries are equal to 1) of some random variables; see section 7.4.

7.4 Necessary and sufficient conditions for covariance and correlation matrices

In Lemma 7.4, we proved that any covariance matrix is symmetric positive semidefinite. We now show, using the Linear Transformation Property, that this is a necessary and sufficient condition by finding normal random variables with covariance matrix equal to any given symmetric positive semidefinite matrix, and establish a similar result for correlation matrices.

Theorem 7.5. *(i) An $n \times n$ square matrix is the covariance matrix of n random variables if and only if the matrix is symmetric positive semidefinite.*

(ii) An $n \times n$ square matrix is the correlation matrix of n random variables if and only if the matrix is symmetric positive semidefinite and has all the entries on the main diagonal equal to 1.

Proof. (i) Recall from Lemma 7.4 that any covariance matrix is symmetric positive semidefinite.

To prove that any symmetric positive semidefinite matrix is a covariance matrix, let A be an $n \times n$ symmetric positive semidefinite matrix. We will find random variables $X_1, X_2, \ldots, X_n$ with covariance matrix $\Sigma_{\mathbf{X}} = A$.

Since the matrix A is symmetric, it follows from Theorem 5.4 that there exists an orthogonal matrix Q and a diagonal matrix Λ such that

$$A \;=\; Q\Lambda Q^t. \tag{7.84}$$

Recall that $\Lambda = \mathrm{diag}(\lambda_i)_{i=1:n}$, where λ_i, $i = 1:n$, are the eigenvalues of A. Note that $\lambda_i \geq 0$, for all $i = 1:n$, since the eigenvalues of a symmetric positive semidefinite matrix are nonnegative; see Theorem 5.6. Let $\Lambda^{1/2}$ be the diagonal matrix given by

$$\Lambda^{1/2} \;=\; \mathrm{diag}\left(\sqrt{\lambda_i}\right)_{i=1:n}. \tag{7.85}$$

Using (1.93), we find that

$$\Lambda^{1/2}\,\Lambda^{1/2} \;=\; \mathrm{diag}\,(\lambda_i)_{i=1:n} \;=\; \Lambda. \tag{7.86}$$

Let

$$M \;=\; Q\,\Lambda^{1/2}. \tag{7.87}$$

Then,

$$M^t \;=\; (\Lambda^{1/2})^t\,Q^t \;=\; \Lambda^{1/2}\,Q^t, \tag{7.88}$$

since $\Lambda^{1/2}$ is a diagonal matrix and therefore symmetric, i.e., $(\Lambda^{1/2})^t = \Lambda^{1/2}$. Then, from (7.87), (7.88), (7.86), and (7.84), we obtain that

$$MM^t \;=\; Q\Lambda^{1/2}\Lambda^{1/2}Q^t \;=\; Q\Lambda Q^t \;=\; A. \tag{7.89}$$

Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal variables, and let $X_1, X_2, \ldots, X_n$ be random variables[4] given by $\mathbf{X} = M\mathbf{Z}$, where M is the matrix given by (7.87), $\mathbf{X} = (X_i)_{i=1:n}$ and $\mathbf{Z} = (Z_i)_{i=1:n}$. Let $\Sigma_{\mathbf{X}}$ be the covariance matrix of $X_1, X_2, \ldots, X_n$. Note that the covariance matrix $\Sigma_{\mathbf{Z}}$ of $Z_1, Z_2, \ldots, Z_n$ is the identity matrix, i.e., $\Sigma_{\mathbf{Z}} = I$. Then, from the Linear Transformation Property (Theorem 7.3) and using (7.89), we find that

$$\Sigma_{\mathbf{X}} \;=\; M\Sigma_{\mathbf{Z}}M^t \;=\; MM^t \;=\; A.$$

In other words, we found random variables $X_1, X_2, \ldots, X_n$ with covariance matrix equal to the matrix A. We conclude that any symmetric positive semidefinite matrix is a covariance matrix.

[4]Note that $X_1, X_2, \ldots, X_n$ are, in fact, normal random variables, since they are linear combinations of independent normal variables.

(ii) Recall from Lemma 7.4 that any correlation matrix is symmetric positive semidefinite with main diagonal entries equal to 1; see (7.11).

To prove that any symmetric positive semidefinite matrix with main diagonal entries equal to 1 is a correlation matrix, let A be an $n \times n$ symmetric positive semidefinite with $A(i,i) = 1$ for all $i = 1 : n$. We will find random variables X_1, X_2, $\ldots$, X_n with correlation matrix $\Omega_{\mathbf{X}} = A$.

We showed above that there exist random variables $X_1, X_2, \ldots, X_n$ with covariance matrix $\Sigma_{\mathbf{X}} = A$. Since $A(i,i) = 1$ for all $i = 1 : n$, it follows that $\Sigma_{\mathbf{X}}(i,i) = 1$ for all $i = 1 : n$, and, from Lemma 7.2, we conclude that $\Omega_{\mathbf{X}} = \Sigma_{\mathbf{X}} = A$, which is what we wanted to show. □

The method for finding normal random variables with a given covariance matrix described above requires finding the eigenvalues of the given symmetric positive semidefinite matrix. Note that this method is not used in practice if the given matrix is symmetric positive definite, in which case the Cholesky decomposition of the given covariance matrix is used; see section 7.5 for details.

Example: The 3×3 matrix

$$A = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \tag{7.90}$$

is a correlation matrix if and only if

$$-1 \leq a, b, c \leq 1 \quad \text{and} \quad \det(A) = 1 + 2abc - a^2 - b^2 - c^2 \geq 0. \tag{7.91}$$

Solution: Since all the main diagonal entries of A are equal to 1, it follows from Theorem 7.5 that the matrix A is a correlation matrix if and only if it is symmetric positive semidefinite, which, according to (5.64) is equivalent to

$$-1 \leq a, b, c \leq 1 \ \text{ and } \ \det(A) = 1 + 2abc - a^2 - b^2 - c^2 \geq 0. \quad \Box \tag{7.92}$$

Example: Find all the values of ρ such that the matrix

$$\Omega = \begin{pmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & \rho \\ 0.3 & \rho & 1 \end{pmatrix}$$

is a correlation matrix.

Solution: From (7.91), it follows that the matrix Ω is a correlation matrix if and only if $-1 \leq \rho \leq 1$ and

$$\det(\Omega) \ = \ 1 + 2 \cdot (0.8) \cdot (0.3) \cdot \rho - \rho^2 - (0.8)^2 - (0.3)^2 \ \geq \ 0,$$

which is equivalent to

$$\rho^2 - 0.48\rho - 0.27 \ \leq \ 0. \tag{7.93}$$

Thus, ρ must be between the roots -0.332364 and 0.812364 of the quadratic equation $\rho^2 - 0.48\rho - 0.27 = 0$ corresponding to (7.93), which is equivalent to $-0.332364 \leq \rho \leq 0.812364$; note that the condition $-1 \leq \rho \leq 1$ is satisfied for all such values of ρ.

We conclude that the matrix Ω is a correlation matrix if and only if

$$-0.332364 \;\leq\; \rho \;\leq\; 0.812364. \quad \Box$$

Example: Show that it is not possible to find three random variables on the same probability space with correlations 0.75, 0.75, and −0.75. In other words, show that it is not possible to find random variables X_1, X_2, X_3 such that

$$\mathrm{corr}(X_1, X_2) = 0.75; \quad \mathrm{corr}(X_1, X_3) = 0.75; \quad \mathrm{corr}(X_2, X_3) = -0.75. \tag{7.94}$$

Solution: We give a proof by contradiction. Assume that random variables X_1, X_2, X_3 with correlations given by (7.94) exist. Then, the correlation matrix of X_1, X_2, X_3 is

$$\Sigma_{\mathbf{X}} \;=\; \begin{pmatrix} 1 & 0.75 & 0.75 \\ 0.75 & 1 & -0.75 \\ -0.75 & -0.75 & 1 \end{pmatrix},$$

which is the same as the matrix from (7.90) with $a = 0.75$, $b = 0.75$, and $c = -0.75$.

However, the condition (7.91) for the matrix $\Sigma_{\mathbf{X}}$ to be a correlation matrix is not satisfied since, for $a = 0.75$, $b = 0.75$, and $c = -0.75$, we obtain that

$$\begin{aligned} 1 + 2abc - a^2 - b^2 - c^2 &= 1 + 2(0.75)(0.75)(-0.75) \\ &\quad\; - (0.75)^2 - (0.75)^2 - (-0.75)^2 \\ &= 1 - 0.84375 - 1.6875 \;=\; -1.53125 \\ &< 0. \end{aligned}$$

We conclude that random variables X_1, X_2, X_3 with correlations given by (7.94) do not exist.

7.5 Finding normal variables with a given covariance or correlation matrix

Finding normal random variables with a given correlation matrix is often needed in practice, e.g., for Monte Carlo simulations; see section 7.5.1. A way to do so based on the Cholesky decomposition and the Linear Transformation Property is presented below.

Theorem 7.6. *(i) Let A be a symmetric positive definite matrix, and let U be the Cholesky factor of A. Let Z_1, Z_2, …, Z_n be independent standard normal variables, and let X_1, X_2, …, X_n be random variables given by*

$$\boldsymbol{X} \;=\; U^t \boldsymbol{Z}, \tag{7.95}$$

where $\boldsymbol{X} = (X_i)_{i=1:n}$ and $\boldsymbol{Z} = (Z_i)_{i=1:n}$. Then, X_1, X_2, …, X_n are normal random variables with covariance matrix $\Sigma_{\boldsymbol{X}}$ equal to the given matrix A, i.e.,

$$\Sigma_{\boldsymbol{X}} \;=\; A. \tag{7.96}$$