8.1. LEAST SQUARES FOR IMPLIED VOLATILITY COMPUTATION

Since the matrix $A'A$ is symmetric positive definite, we obtain that the Hessian $D^2f(x_0)$ is symmetric positive definite, and therefore $x_0$ is a minimum point for the function $f(x)$. We conclude that the point $x_0$ is a global minimum point for $f(x)$, since $x_0$ is the only critical point of $f(x)$.

Thus, the solution to the least squares problem (8.1) is given by (8.15), i.e.,

$$x = (A' A)^{-1} A' y.$$  \hspace{1cm} (8.16)

Note that the numerical value of $x$ from (8.16) is computed by solving the linear system $(A' A) x = A' y$ using the Cholesky solver from Table 6.2, since $A' A$ is a symmetric positive definite matrix; see the pseudocode from Table 8.1 for details.

Table 8.1: Least squares implementation

<table>
<thead>
<tr>
<th>Function Call:</th>
<th>$x = \text{least_squares}(A, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input:</td>
<td>$A = m \times n$ matrix; $m &gt; n$</td>
</tr>
<tr>
<td></td>
<td>$y =$ column vector of size $m$</td>
</tr>
<tr>
<td>Output:</td>
<td>$x =$ solution to $\min</td>
</tr>
<tr>
<td></td>
<td>$x =$ linear_solve_cholesky($A', A' y$);</td>
</tr>
</tbody>
</table>

8.1.1 Least squares for implied volatility computation

Consider a European call or put option$^1$ on an underlying asset whose price is assumed to follow a lognormal model. The implied volatility of the option is the unique value of the volatility parameter $\sigma$ from the lognormal model that makes the Black–Scholes value of the option equal to the market price of the option.

More precisely, if $C_m$ and $P_m$ are the market prices of a European call option and of a European put option, respectively, with strike $K$ and maturity $T$ on an underlying asset with spot price $S$ paying dividends continuously at the rate $q$, and assuming that interest rates are constant and equal to $r$, the implied volatility $\sigma_{imp}$ corresponding to the price $C_m$ is, by definition, the solution $\sigma = \sigma_{imp}$ to

$$C_{BS}(S, K, T, \sigma, r, q) = C_m; \hspace{1cm} (8.17)$$

the implied volatility $\sigma_{imp}$ corresponding to price $P_m$ is the solution $\sigma = \sigma_{imp}$ to

$$P_{BS}(S, K, T, \sigma, r, q) = P_m. \hspace{1cm} (8.18)$$

Here, $C_{BS}(S, K, T, \sigma, r, q)$ and $P_{BS}(S, K, T, \sigma, r, q)$ are the Black–Scholes values of a call option and of a put option given by (10.77–10.80), i.e.,

$$C_{BS}(S, K, T, \sigma, r, q) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2); \hspace{1cm} (8.19)$$

$$P_{BS}(S, K, T, \sigma, r, q) = Ke^{-rT}N(-d_2) - Se^{-qT}N(-d_1). \hspace{1cm} (8.20)$$

$^1$See Section 10.3 for a brief overview of European options.
respectively, where $N(z)$ is the cumulative distribution of the standard normal variable, i.e.,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx,$$

and

$$d_1 = \frac{\ln \left( \frac{S}{P} \right) + \left( r - q + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \sigma \sqrt{T}. \quad (8.21)$$

For any plain vanilla European option, the option price $C_m$ or $P_m$, the maturity $T$, the strike $K$, and the spot price $S$ of the underlying asset are known. However, a continuous dividend yield $q$ for the underlying asset is very rarely quoted in the markets and the interest rate $r$ could be chosen from several different discount curves. As seen below, the least squares method and Put-Call parity can be used to overcome these issues and find the implied volatility using Newton’s method.

As an example of how implied volatilities are computed in practice, consider the snapshot from Table 8.2 of the mid prices on March 9, 2012, of the S&P 500 options (ticker symbol SPX) maturing on December 22, 2012. These options are European options and therefore we can use the Black–Scholes framework. Although not needed for this method, the corresponding spot price of the index was 1,370.

**Table 8.2: Dec 2012 SPX option prices on 3/9/2012**

<table>
<thead>
<tr>
<th>Call Strike</th>
<th>Price</th>
<th>Put Strike</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1175</td>
<td>225.40</td>
<td>P1175</td>
<td>46.60</td>
</tr>
<tr>
<td>C1200</td>
<td>205.55</td>
<td>P1200</td>
<td>51.55</td>
</tr>
<tr>
<td>C1225</td>
<td>186.20</td>
<td>P1225</td>
<td>57.15</td>
</tr>
<tr>
<td>C1250</td>
<td>167.50</td>
<td>P1250</td>
<td>63.30</td>
</tr>
<tr>
<td>C1275</td>
<td>149.15</td>
<td>P1275</td>
<td>70.15</td>
</tr>
<tr>
<td>C1300</td>
<td>131.70</td>
<td>P1300</td>
<td>77.70</td>
</tr>
<tr>
<td>C1325</td>
<td>115.25</td>
<td>P1325</td>
<td>86.20</td>
</tr>
<tr>
<td>C1350</td>
<td>99.55</td>
<td>P1350</td>
<td>95.30</td>
</tr>
<tr>
<td>C1375</td>
<td>84.90</td>
<td>P1375</td>
<td>105.30</td>
</tr>
<tr>
<td>C1400</td>
<td>71.10</td>
<td>P1400</td>
<td>116.55</td>
</tr>
<tr>
<td>C1425</td>
<td>58.70</td>
<td>P1425</td>
<td>129.00</td>
</tr>
<tr>
<td>C1450</td>
<td>47.25</td>
<td>P1450</td>
<td>143.20</td>
</tr>
<tr>
<td>C1500</td>
<td>36.70</td>
<td>P1500</td>
<td>173.95</td>
</tr>
<tr>
<td>C1550</td>
<td>29.25</td>
<td>P1550</td>
<td>210.80</td>
</tr>
<tr>
<td>C1600</td>
<td>22.10</td>
<td>P1600</td>
<td>252.40</td>
</tr>
</tbody>
</table>

Recall that the Put-Call parity states that taking a long position in a European call option and a short position in a European put option with the same strike $K$ and maturity $T$ is equivalent to taking a long position in a forward contract with

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2This method was implemented in 2010 in Bloomberg terminals, providing a tenfold improvement in the accuracy of implied volatility calculations over the prior method.

3The mid price of an option is the average of the bid price and ask price of the option.

4More information on SPX options can be found on the Chicago Board Options Exchange (CBOE); see http://www.cboe.com/products/indexopts/spx_spec.aspx
8.1. LEAST SQUARES FOR IMPLIED VOLATILITY COMPUTATION

delivery price $K$ and maturity $T$, and therefore the following relationship between
the values $C$ and $P$ of the call and put options must hold for no-arbitrage:

$$C - P = Se^{-qT} - Ke^{-rT}. \tag{8.22}$$

Let $F = Se^{(r-q)T}$ be the forward price of the asset at time $T$. Then, $Se^{-qT} = Fe^{-rT}$ and the Put–Call parity (8.22) can be written as

$$C - P = Fe^{-rT} - Ke^{-rT}. \tag{8.23}$$

Denote by $\text{disc} = e^{-rT}$ the discount factor, and let $PVF = Fe^{-rT}$ be the present value of the forward price. Then, (8.23) is the same as

$$C - P = PVF - K \cdot \text{disc}. \tag{8.24}$$

The data from Table 8.2 provides call and put options values for 16 different
strikes. From (8.24), it follows that the values of $PVF$ and $\text{disc}$ can be obtained by
solving a least square problem $y \approx Ax$, see (8.2), with $x =$ \begin{align*} PVF & \quad \text{disc} \end{align*} and with the
following $16 \times 2$ matrix $A$ and the following $16 \times 1$ column vector $y$ corresponding to $C - P$ for each strike:

$$A = \begin{pmatrix} 1 & -1175 \\ 1 & -1200 \\ 1 & -1225 \\ 1 & -1250 \\ 1 & -1275 \\ 1 & -1300 \\ 1 & -1325 \\ 1 & -1350 \\ 1 & -1375 \\ 1 & -1400 \\ 1 & -1425 \\ 1 & -1450 \\ 1 & -1500 \\ 1 & -1550 \\ 1 & -1575 \\ 1 & -1600 \end{pmatrix}; \quad y = \begin{pmatrix} 178.80 \\ 154.00 \\ 129.05 \\ 104.20 \\ 79.00 \\ 54.00 \\ 29.05 \\ 4.25 \\ -20.40 \\ -45.45 \\ -70.30 \\ -95.95 \\ -144.70 \\ -195.00 \\ -219.80 \\ -244.50 \end{pmatrix}.$$  

The solution $x = (A^tA)^{-1}A^ty$ to this least squares problem, see (8.16), is computed as $x = \text{least squares}(A, y)$ by using the routine from Table 8.1. We find that

$$x = \begin{pmatrix} PVF \\ \text{disc} \end{pmatrix} = \begin{pmatrix} 1349.54 \\ 0.9964 \end{pmatrix}. \tag{8.25}$$

In order to use the values $PVF = 1349.54$ and $\text{disc} = 0.9964$ obtained above to compute implied volatilities, we first show that the Black–Scholes formulas (8.19–8.21) can be written in terms of $PVF$ and $\text{disc}$ without any dependence on $r$, $q$, or the spot price $S$; see (8.34–8.36).

Recall that $PVF = Fe^{-rT}$. Since $F = Se^{(r-q)T}$, it follows that

$$PVF = Fe^{-rT} = Se^{(r-q)T} \cdot e^{-rT}$$

$$= Se^{-qT}. \tag{8.26}$$
Also, recall that
disc = \(e^{-rT}\), \quad (8.27)

Then, using (8.26) and (8.27), the Black–Scholes formulas (8.19) and (8.20) can be written as

\[ C_{BS} = PVF \cdot N(d_1) - K \cdot disc \cdot N(d_2); \quad (8.28) \]

\[ P_{BS} = K \cdot disc \cdot N(-d_2) - PVF \cdot N(-d_1). \quad (8.29) \]

Moreover,

\[ \ln \left( \frac{S}{K} \right) + (r-q)T = \ln \left( \frac{S}{K} \right) + \ln \left( e^{(r-q)T} \right) = \ln \left( \frac{S}{K} \cdot e^{(r-q)T} \right) \quad (8.30) \]

\[ = \ln \left( \frac{PVF}{K \cdot disc} \right) \quad (8.31) \]

where for (8.30) we used the facts that \(\ln(e^x) = x\) for any \(x\) and \(\ln(a) + \ln(b) = \ln(ab)\) for any \(a, b > 0\), and for (8.31) we used (8.26) and (8.27), i.e., \(Se^{-qT} = PVF\) and \(e^{-rT} = disc\).

Then, from (8.21) and (8.31), we obtain that

\[ d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r-q)T}{\sigma \sqrt{T}} + \frac{\sigma^2 T}{2} = \frac{\ln \left( \frac{PVF}{disc} \right)}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2}; \quad (8.32) \]

\[ d_2 = d_1 - \sigma \sqrt{T} = \frac{\ln \left( \frac{PVF}{disc} \right)}{\sigma \sqrt{T}} - \frac{\sigma \sqrt{T}}{2}. \quad (8.33) \]

From (8.28), (8.29), (8.32), and (8.33), we conclude that the Black–Scholes option values can be written as functions of \(PVF, disc, K, T,\) and \(\sigma\) as follows:

\[ C_{BS}(PVF, disc, K, T, \sigma) = PVF \cdot N(d_1) - K \cdot disc \cdot N(d_2); \quad (8.34) \]

\[ P_{BS}(PVF, disc, K, T, \sigma) = K \cdot disc \cdot N(-d_2) - PVF \cdot N(-d_1), \quad (8.35) \]

where

\[ d_1 = \frac{\ln \left( \frac{PVF}{disc} \right)}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2}; \quad d_2 = \frac{\ln \left( \frac{PVF}{disc} \right)}{\sigma \sqrt{T}} - \frac{\sigma \sqrt{T}}{2}. \quad (8.36) \]

Since \(K\) and \(T\) are known and \(PVF\) and \(disc\) have been computed using least squares, see (8.25), we can use Newton’s method to solve either

\[ C_{BS}(PVF, disc, T, \sigma) = C_m \quad (8.37) \]

or

\[ P_{BS}(PVF, disc, T, \sigma) = P_m \quad (8.38) \]

for \(\sigma = \sigma_{imp}\) for every option from Table 8.2.

For call options, we look at (8.37) as a function of only one variable, \(\sigma\). Then, finding the implied volatility for a call option requires solving the nonlinear problem

\[ f_C(x) = 0, \quad (8.39) \]
using Newton’s method, where \( x = \sigma \) and

\[
f_C(x) = PVF \cdot N(d_1(x)) - K \cdot disc \cdot N(d_2(x)) - C_m, \tag{8.40}
\]

with \( d_1(x) \) and \( d_2(x) \) given by (8.36), i.e.,

\[
d_1(x) = \frac{\ln \left( \frac{PVF}{K \cdot disc} \right)}{x \sqrt{T}} + \frac{x \sqrt{T}}{2}; \quad d_2(x) = \frac{\ln \left( \frac{PVF}{K \cdot disc} \right)}{x \sqrt{T}} - \frac{x \sqrt{T}}{2}.
\]

The value of \( x \) thus computed is the implied volatility \( \sigma_{imp} \).

Note that differentiating the function \( f_C(x) \) with respect to \( x \) is the same as computing the vega of the call option, which is equal to

\[
\text{vega}_C = S e^{-qT} \sqrt{\frac{T}{2\pi}} \exp \left( -\frac{(d_1(x))^2}{2} \right); \tag{8.41}
\]

see (10.85). Then, since \( S e^{-qT} = PVF \), we obtain that

\[
f'_C(x) = PVF \sqrt{\frac{T}{2\pi}} \exp \left( -\frac{(d_1(x))^2}{2} \right). \tag{8.42}
\]

The Newton’s method recursion for solving (8.39) is

\[
x_{k+1} = x_k - \frac{f_C(x_k)}{f'_C(x_k)} \tag{8.43}
\]

where the functions \( f_C(x) \) and \( f'_C(x) \) are given by (8.40) and (8.42), respectively.

Similarly, for put options we look at (8.38) as a function of only one variable, \( \sigma \). Then, finding the implied volatility for a put option requires solving the nonlinear problem

\[
f_P(x) = 0, \tag{8.44}
\]

where \( x = \sigma \) and

\[
f_P(x) = K \cdot disc \cdot N(-d_2(x)) - PVF \cdot N(-d_1(x)) - P_m, \tag{8.45}
\]

with \( d_1(x) \) and \( d_2(x) \) given by (8.36), i.e.,

\[
d_1(x) = \frac{\ln \left( \frac{PVF}{K \cdot disc} \right)}{x \sqrt{T}} + \frac{x \sqrt{T}}{2}; \quad d_2(x) = \frac{\ln \left( \frac{PVF}{K \cdot disc} \right)}{x \sqrt{T}} - \frac{x \sqrt{T}}{2}.
\]

Differentiating the function \( f_P(x) \) with respect to \( x \) is the same as computing the vega of the put option, which is equal to

\[
\text{vega}_P = S e^{-qT} \sqrt{\frac{T}{2\pi}} \exp \left( -\frac{(d_1(x))^2}{2} \right), \tag{8.46}
\]

respectively; see (10.86) and (10.85).

Then, since \( S e^{-qT} = PVF \), we obtain that

\[
f'_P(x) = PVF \sqrt{\frac{T}{2\pi}} \exp \left( -\frac{(d_1(x))^2}{2} \right). \tag{8.47}
\]
Note that $f_P(x) = f_C(x)$, since vega$_P = $ vega$_C$.

The Newton’s method recursion for solving (8.44) is

$$x_{k+1} = x_k - \frac{f_P(x_k)}{f'_P(x_k)},$$

(8.48)

where the functions $f_P(x)$ and $f'_P(x)$ are given by (8.45) and (8.47), respectively.

A good initial guess for Newton’s method is 25% volatility, i.e., $x_0 = 0.25$, and the algorithm is stopped when two consecutive approximations in Newton’s method are within $10^{-6}$ of each other; see the pseudocode from Table 8.3 for finding the implied volatility for both call and put options, i.e., for solving either (8.43) or (8.48).

Table 8.3: Pseudocode for computing implied volatility

<table>
<thead>
<tr>
<th>Input:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_m =$ option price</td>
</tr>
<tr>
<td>$// V_m = C_m$ for call implied vol; $V_m = P_m$ for put implied vol</td>
</tr>
<tr>
<td>$K =$ strike price of the option</td>
</tr>
<tr>
<td>$T =$ maturity of the option</td>
</tr>
<tr>
<td>$PVF =$ present value of the forward price of the underlying asset</td>
</tr>
<tr>
<td>$disc =$ discount factor corresponding to time $T$</td>
</tr>
<tr>
<td>$tol =$ tolerance for Newton’s method convergence</td>
</tr>
<tr>
<td>$f_{BS}(x) =$ Black–Scholes option value; $x =$ volatility</td>
</tr>
<tr>
<td>$// f_{BS}(x) = f_C(x)$ for calls; $f_{BS}(x) = f_P(x)$ for puts</td>
</tr>
<tr>
<td>$// f'_{BS}(x) = f'_C(x) = f'_P(x)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Output:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{new} =$ implied volatility</td>
</tr>
<tr>
<td>$x_0 = 0.25; // initial guess: 25% volatility</td>
</tr>
<tr>
<td>$x_{new} = x_0; x_{old} = x_0 - 1; tol = 10^{-6}$</td>
</tr>
<tr>
<td>while $</td>
</tr>
<tr>
<td>$x_{old} = x_{new}$</td>
</tr>
<tr>
<td>$x_{new} = x_{old} - \frac{f_{BS}(x_{old}) - V_m}{f'<em>{BS}(x</em>{old})}$</td>
</tr>
<tr>
<td>end</td>
</tr>
</tbody>
</table>

For the options from Table 8.2, note that the options maturity is $T = \frac{199}{252}$, i.e., the ratio of 199, the number of trading days between March 9, 2012 and December 22, 2012, and 252, the total number of trading days in a year. Using the values $PVF = 1349.54$ and $disc = 0.9964$ computed using least squares and the Newton’s method from Table 8.3, we obtain the implied volatilities from Table 8.4.

A consequence of the Put–Call parity is that the theoretical values of implied volatilities of calls and puts with the same strike are equal. Note that, indeed, the implied volatilities from Table 8.4 corresponding to calls and puts with the same strike are nearly identical.
8.2 Linear regression: ordinary least squares for time series data

Linear regression for time series data (also called ordinary least squares for time series data) requires finding the best approximation of the time series data of a random variable by a linear combination of the time series data of other random variables and a constant vector.

Let $Y$ and $X_1, X_2, \ldots, X_n$ be random variables given by time series data at $N$ data points $t_i, i = 1 : N$. Assume that the sample covariance matrix corresponding to the time series data for $X_1, X_2, \ldots, X_n$ is nonsingular, or, equivalently, that the column vectors of the time series data for $X_1, X_2, \ldots, X_n$ and the $N \times 1$ column vector with all entries equal to 1 are linearly independent; cf. Theorem 7.2.

We look for the best linear approximation of the time series data for $Y$ by a linear combination of the time series data for $X_1, X_2, \ldots, X_n$ plus a constant vector, i.e., we look for constants $a, b_1, b_2, \ldots, b_n$ such that

$$
\begin{pmatrix}
Y(t_1) \\
Y(t_2) \\
\vdots \\
Y(t_N)
\end{pmatrix}
\approx
\begin{pmatrix}
a + \sum_{k=1}^n b_k X_k(t_1) \\
a + \sum_{k=1}^n b_k X_k(t_2) \\
\vdots \\
a + \sum_{k=1}^n b_k X_k(t_N)
\end{pmatrix}
$$

$$
\begin{pmatrix}
Y(t_1) \\
Y(t_2) \\
\vdots \\
Y(t_N)
\end{pmatrix}
\approx
\begin{pmatrix}
a \\
a \\
\vdots \\
\vdots \\
a
\end{pmatrix}
+ \sum_{k=1}^n b_k \begin{pmatrix} X_k(t_1) \\ X_k(t_2) \\ \vdots \\ X_k(t_N) \end{pmatrix}
$$

$$
\begin{pmatrix}
Y(t_1) \\
Y(t_2) \\
\vdots \\
Y(t_N)
\end{pmatrix}
\approx
\begin{pmatrix}
1 & X_1(t_1) & \ldots & X_n(t_1) \\
1 & X_1(t_2) & \ldots & X_n(t_2) \\
\vdots & \vdots & \ddots & \vdots \\
1 & X_1(t_N) & \ldots & X_n(t_N)
\end{pmatrix}
\begin{pmatrix}
a \\
b_1 \\
\vdots \\
b_n
\end{pmatrix}
$$

(8.49)

Denote by $T_Y$ and $TX_k$ the column vectors of the time series data for the random variables $Y$ and $X_k$, for $k = 1 : n$, respectively, i.e.,

$$
T_Y = (Y(t_i))_{i=1:N};
$$

(8.50)