

A LINEAR ALGEBRA PRIMER

for

FINANCIAL ENGINEERING

Covariance Matrices, Eigenvectors, OLS,
and more

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