Chapter 1

Vectors and matrices.

1.1 Exercises

1. Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 5 & 4 \\ 3 & -2 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 3 \\ -5 & 4 & 2 & -4 & 3 \end{pmatrix}.$$

Show that the column rank and the row rank of A are both equal to 3.

2. Let x and y be column vectors of size n, and let I be the identity matrix of size n.

(i) If $y^t x \neq -1$, show that

$$(I + xy^t)^{-1} = I - \frac{1}{1 + y^t x} xy^t.$$

In other words, show that

$$\left(I - \frac{1}{1 + y^t x} xy^t \right) (I + xy^t) = I.$$

(ii) Show that the matrix $I + xy^t$ is nonsingular if and only if $y^t x \neq -1$.

3. (i) Use induction to show that

$$\left(\prod_{i=1}^{n} A_i \right)^t = \prod_{i=1}^{n} A_{n+1-i}^t$$

for any $m_i \times n_i$ matrices A_i, $i = 1 : n$, with $n_i = m_{i+1}$ for $i = 1 : (n-1)$.

(ii) Show that

$$\left(\prod_{i=1}^{n} A_i \right)^{-1} = \prod_{i=1}^{n} A_{n+1-i}^{-1}$$

for any nonsingular square matrices A_i of the same size.

4. Let $D = \text{diag}(d_i)_{i=1:n}$ be a diagonal matrix of size n with distinct diagonal entries, i.e., such that $d_j \neq d_k$, for any $1 \leq j \neq k \leq n$. If A is a square matrix of size n, show that $AD = DA$ if and only if the matrix A is diagonal.

5. Use the fact that $D_1 D_2 = D_2 D_1$ for any two diagonal matrices D_1 and D_2 of the same size to show that

$$\prod_{i=1}^{n} D_i \;=\; \prod_{i=1}^{n} D_{p(i)},$$

for any one-to-one function $p : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$, where D_i, $i = 1 : n$, are diagonal matrices of the same size.

6. (i) Let A be an $n \times n$ matrix and let L be an $n \times n$ nonsingular lower triangular matrix. Show that, if LA is a lower triangular matrix, then A is lower triangular. Show that, if AL is a lower triangular matrix, then A is lower triangular.

   (ii) Let A be an $n \times n$ matrix and let U be an $n \times n$ nonsingular upper triangular matrix. Show that, if UA is an upper triangular matrix, then A is upper triangular. Show that, if AU is an upper triangular matrix, then A is upper triangular.

7. Let A be a nonsingular matrix, and let k be a positive integer. Define A^{-k} as the k–th power of the inverse matrix of A, i.e., let $A^{-k} = \left(A^{-1}\right)^k$. Show that this definition is consistent, i.e., show that

$$A^k \cdot A^{-k} \;=\; A^{-k} \cdot A^k \;=\; I.$$

8. (i) Let

$$M \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix}.$$

   Compute M^2, M^3, M^4.

   (ii) Let

$$C \;=\; I + M \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{pmatrix}.$$

   Compute C^m, where $m \geq 2$ is a positive integer.

   Hint: Recall that, if A and B are square matrices of the same size such that $AB = BA$, then the following version of the binomial formula holds true:

$$(A + B)^m \;=\; \sum_{j=0}^{m} \binom{m}{j} A^j B^{m-j}, \tag{1.1}$$

where m is a positive integer and the binomial coefficient $\left(\begin{array}{c} m \\ j \end{array}\right)$ is given by

$$\left(\begin{array}{c} m \\ j \end{array}\right) = \frac{m!}{j!\,(m-j)!},$$

where $k! = 1 \cdot 2 \cdot \ldots \cdot k$. Also, note that $A^0 = B^0 = I$.

9. Let L be an $n \times n$ lower triangular matrix with entries equal to 0 on the main diagonal, i.e., with $L(i,i) = 0$ for $i = 1 : n$.

   (i) Show that $L^n = 0$;

   (ii) Compute $(I + L)^m$ in terms of L, L^2, $\ldots$, L^{n-1}, where $m \geq n$ is a positive integer.

   Hint: Use the binomial formula (1.1).

10. Let A and B be square matrices of the same size with nonnegative entries and such that the sum of the entries in each row is equal to 1. Show that the matrix AB has the same properties, i.e., show that all the entries of the matrix AB are nonnegative and the sum of the entries in each row of AB is equal to 1.

    Note: A matrix with nonnegative entries such that the sum of the entries in each row is equal to 1 is called a probability matrix.

11. The covariance matrix of five random variables is

$$\Sigma = \begin{pmatrix} 1 & -0.525 & 1.375 & -0.075 & -0.75 \\ -0.525 & 2.25 & 0.1875 & 0.1875 & -0.675 \\ 1.375 & 0.1875 & 6.25 & 0.4375 & -1.875 \\ -0.075 & 0.1875 & 0.4375 & 0.25 & 0.3 \\ -0.75 & -0.675 & -1.875 & 0.3 & 9 \end{pmatrix}.$$

    Find the correlation matrix of these random variables.

12. The correlation matrix of five random variables is

$$\Omega = \begin{pmatrix} 1 & -0.25 & 0.15 & -0.05 & -0.30 \\ -0.25 & 1 & -0.10 & -0.25 & 0.10 \\ 0.15 & -0.10 & 1 & 0.20 & 0.05 \\ -0.05 & -0.25 & 0.20 & 1 & 0.10 \\ -0.30 & 0.10 & 0.05 & 0.10 & 1 \end{pmatrix}$$

    (i) Compute the covariance matrix of these random variables if their standard deviations are 0.25, 0.5, 1, 2, and 4, in this order.

    (ii) Compute the covariance matrix of these random variables if their standard deviations are 4, 2, 1, 0.5, and 0.25, in this order.

13. The file *indeces-jul26-aug9-2012.xlsx* from fepress.org/nla-primer contains the July 26, 2012 – August 9, 2012 end of day values of Dow Jones, Nasdaq, and S&P 500.

    (i) Compute the daily percentage returns of the three indices over the given time period.

    (ii) Compute the covariance matrix of the daily percentage returns of the three indices.

    (iii) Compute the daily log returns of the three indices over the given time period.

    (iv) Compute the covariance matrix of the daily log returns of the three indices.

    Note: The percentage return and the log return between times t_1 and t_2 of an asset with price $S(t)$ at time t are given by

    $$\frac{S(t_2) - S(t_1)}{S(t_1)} \quad \text{and} \quad \ln\left(\frac{S(t_2)}{S(t_1)}\right),$$

    respectively.

14. The file *indices-july2011.xlsx* from fepress.org/nla-primer contains the January 2011 – July 2011 end of day values of nine major US indices.

    (i) Compute the sample covariance matrix of the daily percentage returns of the indices, and the corresponding sample correlation matrix.

    Compute the sample covariance and correlation matrices for daily log returns, and compare them with the corresponding matrices for daily percentage returns.

    (ii) Compute the sample covariance matrix of the weekly percentage returns of the indices, and the corresponding sample correlation matrix.

    Compute the sample covariance and correlation matrices for weekly log returns, and compare them with the corresponding matrices for weekly percentage returns.

    (iii) Compute the sample covariance matrix of the monthly percentage returns of the indices, and the corresponding sample correlation matrix.

    Compute the sample covariance and correlation matrices for monthly log returns, and compare them with the corresponding matrices for monthly percentage returns.

    (iv) Comment on the differences between the sample covariance and correlation matrices for daily, weekly, and monthly returns.

15. In three months, the value of an asset with spot price \$50 will be either \$60 or \$45. The continuously compounded risk–free rate is 6%. Consider the one period market model with two securities, i.e., cash and the asset, and two states, i.e., asset value equal to \$60 and asset value equal to \$45, in three months.

(i) Find the payoff matrix of this model.

(ii) Is this one period market complete, i.e., is the payoff matrix nonsingular?

(iii) How do you replicate a three months at–the–money put option on this asset, using the cash and the underlying asset?

16. In six months, the price of an asset with spot price \$40 will be either \$30, \$35, \$40, \$42, \$45, or \$50. Consider a one period market model with six states in six months corresponding to the six possible values of the asset in six months, and with the following four securities:

- cash;
- asset;
- six months at-the-money call option with strike \$40 on the asset;
- six months at-the-money put option with strike \$40 on the asset.

The continuously compounded risk–free interest rate is constant and equal to 6%.

(i) Find the payoff matrix of this model.

(ii) Is this one period market model complete?

(iii) Are the four securities non–redundant?