1.2 Solutions to Chapter 1 Exercises

Problem 1: Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 5 & 4 \\ 3 & -2 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 3 \\ -5 & 4 & 2 & -4 & 3 \end{pmatrix}.$$

Show that the column rank and the row rank of A are both equal to 3.

Solution: Let

$$(c_1 \mid c_2 \mid c_3 \mid c_4 \mid c_5) = \begin{pmatrix} 1 & -1 & 2 & 5 & 4 \\ 3 & -2 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 3 \\ -5 & 4 & 2 & -4 & 3 \end{pmatrix}$$

be the column form of the matrix A. By doing column reduction for A, we obtain that

$$\begin{aligned} & (c_1 \mid c_1 + c_2 \mid c_3 - 2c_1 \mid c_4 - 5c_1 \mid c_5 - 4c_1) \\ = & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & -5 & -11 & -10 \\ 0 & 1 & 2 & -1 & 3 \\ -5 & -1 & 12 & 21 & 23 \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} & (c_1 \mid c_1 + c_2 \mid c_3 - 2c_1 + 5(c_1 + c_2) \mid c_4 - 5c_1 + 11(c_1 + c_2) \mid \\ & \qquad c_5 - 4c_1 + 10(c_1 + c_2)) \\ = & (c_1 \mid c_1 + c_2 \mid 3c_1 + 5c_2 + c_3 \mid 6c_1 + 11c_2 + c_4 \mid 6c_1 + 10c_2 + c_5) \qquad (1.2) \\ = & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 10 & 13 \\ -5 & -1 & 7 & 10 & 13 \end{pmatrix}. \qquad (1.3) \end{aligned}$$

From (1.2) and (1.3), it follows that the first three columns c_1, c_2, and c_3 of the matrix A are linearly independent, while

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{7}(3c_1 + 5c_2 + c_3) = \frac{1}{10}(6c_1 + 11c_2 + c_4) = \frac{1}{13}(6c_1 + 10c_2 + c_5).$$

Then,

$$\frac{1}{10}(6c_1 + 11c_2 + c_4) = \frac{1}{7}(3c_1 + 5c_2 + c_3) \iff c_4 = -\frac{12}{7}c_1 - \frac{27}{7}c_2 + \frac{10}{7}c_3;$$

$$\frac{1}{13}(6c_1 + 10c_2 + c_5) = \frac{1}{7}(3c_1 + 5c_2 + c_3) \iff c_5 = -\frac{3}{7}c_1 - \frac{5}{7}c_2 + \frac{13}{7}c_3,$$

and therefore the columns c_4 and c_5 are linearly dependent on c_1, c_2, c_3.

We conclude that the column rank of the matrix A is 3.

Let

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 5 & 4 \\ 3 & -2 & 1 & 4 & 2 \\ 0 & 1 & 2 & -1 & 3 \\ -5 & 4 & 2 & -4 & 3 \end{pmatrix}$$

be the row form of the matrix A. By doing row reduction for A, we obtain that

$$\begin{pmatrix} r_1 \\ r_2 - 3r_1 \\ r_3 \\ r_4 + 5r_1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & 5 & 4 \\ 0 & 1 & -5 & -11 & -10 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & -1 & 12 & 21 & 23 \end{pmatrix};$$

$$\begin{pmatrix} r_1 \\ r_2 - 3r_1 \\ r_3 - (r_2 - 3r_1) \\ r_4 + 5r_1 + (r_2 - 3r_1) \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 - 3r_1 \\ 3r_1 - r_2 + r_3 \\ 2r_1 + r_2 + r_4 \end{pmatrix} \tag{1.4}$$

$$= \begin{pmatrix} 1 & -1 & 2 & 5 & 4 \\ 0 & 1 & -5 & -11 & -10 \\ 0 & 0 & 7 & 10 & 13 \\ 0 & 0 & 7 & 10 & 13 \end{pmatrix}. \tag{1.5}$$

From (1.4) and (1.5), it follows that the first three rows r_1, r_2, and r_3 of the matrix A are linearly independent, while

$$3r_1 - r_2 + r_3 = 2r_1 + r_2 + r_4 = (0 \ \ 0 \ \ 7 \ \ 10 \ \ 13).$$

Then,

$$r_4 = r_1 - 2r_2 + r_3,$$

and therefore the row r_4 is linearly dependent on r_1, r_2, r_3.

We conclude that the row rank of the matrix A is 3. □

Problem 2: Let x and y be column vectors of size n, and let I be the identity matrix of size n.

(i) If $y^t x \neq -1$, show that

$$(I + xy^t)^{-1} = I - \frac{1}{1 + y^t x} \, xy^t. \tag{1.6}$$

In other words, show that

$$\left(I - \frac{1}{1 + y^t x} \, xy^t \right) (I + xy^t) = I. \tag{1.7}$$

(ii) Show that the matrix $I + xy^t$ is nonsingular if and only if $y^t x \neq -1$.

Solution: (i) Note that the right hand side of (1.6) in indeed an $n \times n$ matrix: xy^t is the result of a column vector–row vector multiplication, and therefore it is an $n \times n$ matrix, while $y^t x$ is the result of a row vector–column vector multiplication, and therefore it is a number, which means $\frac{1}{1+y^t x}$ is a number.

Then,

$$\begin{aligned}
\left(I - \frac{1}{1+y^t x}\, xy^t\right)(I + xy^t) &= I + xy^t - \frac{1}{1+y^t x}\, xy^t(I + xy^t) \\
&= I + xy^t - \frac{1}{1+y^t x}\, xy^t - \frac{1}{1+y^t x}\, (xy^t)(xy^t) \\
&= I + xy^t - \frac{1}{1+y^t x}\, xy^t - \frac{1}{1+y^t x}\, x(y^t x)y^t \\
&= I + xy^t - \frac{1}{1+y^t x}\, xy^t - \frac{y^t x}{1+y^t x}\, xy^t \qquad (1.8) \\
&= I + xy^t - \frac{1+y^t x}{1+y^t x}\, xy^t \\
&= I + xy^t - xy^t \\
&= I,
\end{aligned}$$

where, for (1.8), we used the fact that $y^t x$ is a number (since it is the result of a row vector–column vector multiplication), and therefore

$$\frac{1}{1+y^t x}\, x(y^t x)y^t = \frac{1}{1+y^t x}(y^t x)\, xy^t = \frac{y^t x}{1+y^t x}\, xy^t.$$

Thus, (1.7) is established, and therefore $I - \frac{1}{1+y^t x}\, xy^t$ is the inverse matrix of $I + xy^t$, i.e.,

$$(I + xy^t)^{-1} = I - \frac{1}{1+y^t x}\, xy^t.$$

(ii) To show that the matrix $I + xy^t$ is nonsingular if and only if $y^t x \neq -1$, note that we showed in part (i) that, if $y^t x \neq -1$, then the matrix $I + xy^t$ has an inverse and it is therefore nonsingular.

Thus, we only need to show that, if $y^t x = -1$, then the matrix $I + xy^t$ is singular.

Assume that $y^t x = -1$. Then,

$$(I + xy^t)x = x + xy^t x = x + x(y^t x) = x + x(-1) = x - x = 0.$$

Thus,

$$(I + xy^t)x = 0. \qquad (1.9)$$

If the matrix $I + xy^t$ were nonsingular, then we multiply (1.9) to the left by the inverse of the matrix $I + xy^t$ and obtain that

$$\begin{aligned}
(I + xy^t)x = 0 \quad &\Longleftrightarrow \quad (I + xy^t)^{-1}(I + xy^t)x = 0 \\
&\Longleftrightarrow \quad x = 0,
\end{aligned}$$

since $(I + xy^t)^{-1}(I + xy^t) = I$.

However, if $x = 0$, then $y^t x = 0$, which is not possible since $y^t x = -1$. This is a contradiction which comes from the assumption that the matrix $I + xy^t$ is nonsingular.

We conclude that, if $y^t x = -1$, then the matrix $I + xy^t$ is singular, which is what we wanted to show. □

Problem 3: (i) Use induction to show that

$$\left(\prod_{i=1}^{n} A_i\right)^t = \prod_{i=1}^{n} A_{n+1-i}^t \tag{1.10}$$

for any $m_i \times n_i$ matrices A_i, $i = 1 : n$, with $n_i = m_{i+1}$ for $i = 1 : (n-1)$.

(ii) Show that

$$\left(\prod_{i=1}^{n} A_i\right)^{-1} = \prod_{i=1}^{n} A_{n+1-i}^{-1} \tag{1.11}$$

for any nonsingular square matrices A_i of the same size.

Solution: (i) Recall that, if B is an $m \times n$ matrix and C is an $n \times p$ matrix, than

$$(BC)^t = C^t B^t. \tag{1.12}$$

Note that (1.10) can be written as

$$(A_1 A_2 \ldots A_{n-1} A_n)^t = A_n^t A_{n-1}^t \ldots A_2^t A_1^t. \tag{1.13}$$

We will prove (1.13) by induction.

For $n = 2$, (1.13) becomes $(A_1 A_2)^t = A_2^t A_1^t$, which holds true; see (1.12).

Assume that (1.13) holds true for n, i.e., assume that

$$(A_1 A_2 \ldots A_{n-1} A_n)^t = A_n^t A_{n-1}^t \ldots A_2^t A_1^t. \tag{1.14}$$

We will show that (1.13) holds true for $n+1$, i.e.,

$$(A_1 A_2 \ldots A_n A_{n+1})^t = A_{n+1}^t A_n^t \ldots A_2^t A_1^t. \tag{1.15}$$

Using (1.12) for $B = A_1 A_2 \ldots A_n$ and $C = A_{n+1}$, and using the induction hypothesis (1.14), we find that

$$\begin{aligned} (A_1 A_2 \ldots A_n A_{n+1})^t &= ((A_1 A_2 \ldots A_n) \cdot A_{n+1})^t \\ &= A_{n+1}^t (A_1 A_2 \ldots A_n)^t \\ &= A_{n+1}^t A_n^t \ldots A_2^t A_1^t, \end{aligned}$$

which is the same as (1.15).

We conclude that (1.13), which is the same as (1.10), is proved by induction.

(ii) The proof of (1.11) follows the exact same steps as the proof of (1.10) from (i) and is based on the fact that, if B and C are nonsingular matrices of the same size, then $(BC)^{-1} = C^{-1}B^{-1}$, which mirrors the property (1.12) for matrix transposes. □

Problem 4: Let $D = \operatorname{diag}(d_i)_{i=1:n}$ be a diagonal matrix of size n with distinct diagonal entries, i.e., such that $d_j \neq d_k$, for any $1 \leq j \neq k \leq n$. If A is a square matrix of size n, show that $AD = DA$ if and only if the matrix A is diagonal.

Solution: Let $A = \mathrm{col}\,(a_k)_{k=1:n}$, with $a_k = (A(j,k))_{j=1:n}$, be the column form of the matrix A, and let $A = \mathrm{row}\,(r_j)_{j=1:n}$, with $r_j = (A(j,k))_{k=1:n}$, be the row form of A. Recall that

$$AD \;=\; \mathrm{col}\,(d_k a_k)_{k=1:n} \;=\; (d_1 a_1 \mid d_2 a_2 \mid \ldots \mid d_n a_n)\,; \tag{1.16}$$

$$DA \;=\; \mathrm{row}\,(d_j r_j)_{j=1:n} \;=\; \begin{pmatrix} d_1 r_1 \\ d_2 r_2 \\ \vdots \\ d_n r_n \end{pmatrix}. \tag{1.17}$$

From (1.16) and (1.17), we find

$$(AD)(j,k) \;=\; d_k A(j,k), \quad \forall\ 1 \le j,k \le n; \tag{1.18}$$

$$(DA)(j,k) \;=\; d_j A(j,k), \quad \forall\ 1 \le j,k \le n. \tag{1.19}$$

From (1.18) and (1.19), we conclude that

$$\begin{aligned} AD = DA \;&\Longleftrightarrow\; (AD)(j,k) = (DA)(j,k), \quad \forall\ 1 \le j,k \le n \\ &\Longleftrightarrow\; d_k A(j,k) = d_j A(j,k), \quad \forall\ 1 \le j,k \le n \\ &\Longleftrightarrow\; (d_k - d_j) A(j,k) = 0, \quad \forall\ 1 \le j,k \le n \\ &\Longleftrightarrow\; A(j,k) = 0, \quad \forall\ 1 \le j \ne k \le n, \end{aligned} \tag{1.20}$$

where (1.20) comes from the fact that $d_j \ne d_k$ for all $1 \le j \ne k \le n$, and therefore

$$d_k - d_j \ne 0, \quad \forall\ 1 \le j \ne k \le n.$$

In other words, $AD = DA$ if and only if $A(j,k) = 0$ for all $1 \le j \ne k \le n$, i.e., if and only if the matrix A is diagonal. □

Problem 5: Use the fact that $D_1 D_2 = D_2 D_1$ for any two diagonal matrices D_1 and D_2 of the same size to show that

$$\prod_{i=1}^{n} D_i \;=\; \prod_{i=1}^{n} D_{p(i)}, \tag{1.21}$$

for any one-to-one function $p : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$, where D_i, $i = 1 : n$, are diagonal matrices of the same size.

Solution: The intuition behind the proof of this result is that, given a product of n diagonal matrices of the same size, we can move the matrix D_n to the last position in the product without changing the product, by using the fact that the product of two diagonal matrices is commutative. Then, we use the same commutativity property to move the matrix D_{n-1} to the $n-1$ position, followed by moving the matrix D_{n-2} to the $n-2$ position, and so on, until moving matrix D_2 to position 2, which leaves matrix D_1 in the first position, thus concluding that the product of n diagonal matrices, regardless of the initial order, is $D_1 D_2 \ldots D_n$.

A formal proof of (1.21) based on the idea above can be given by induction.

Assume that

$$\prod_{i=1}^{n} D_i = \prod_{i=1}^{n} D_{p_n(i)}, \tag{1.22}$$

for any one-to-one function $p_n : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$, where D_i, $i = 1 : n$, are diagonal matrices of the same size.

Let $p_{n+1} : \{1, 2, \ldots n, n+1\} \to \{1, 2, \ldots n, n+1\}$ be a one-to-one function. We will show that

$$\prod_{i=1}^{n+1} D_i = \prod_{i=1}^{n+1} D_{p_{n+1}(i)}, \tag{1.23}$$

where D_i, $i = 1 : n+1$, are diagonal matrices of the same size.

Since p_{n+1} is a one-to-one function from a set of $n+1$ elements to a set of $n+1$ elements, it follows that p_{n+1} is also an onto function, and therefore there exists l with $1 \le l \le n+1$ such that $p_{n+1}(l) = n+1$. Then,

$$\begin{aligned} \prod_{i=1}^{n+1} D_{p_{n+1}(i)} &= \prod_{i=1}^{l-1} D_{p_{n+1}(i)} \cdot D_{p_{n+1}(l)} \cdot \prod_{i=l+1}^{n+1} D_{p_{n+1}(i)} \\ &= \prod_{i=1}^{l-1} D_{p_{n+1}(i)} \cdot D_{n+1} \cdot \prod_{i=l+1}^{n+1} D_{p_{n+1}(i)}, \end{aligned} \tag{1.24}$$

since $p_{n+1}(l) = n+1$.

Note that $\prod_{i=l+1}^{n+1} D_{p_{n+1}(i)}$ is the product of $n - l + 1$ diagonal matrices and therefore it is a diagonal matrix. Since the product of any two diagonal matrices is a commutative, it follows that

$$\cdot D_{n+1} \cdot \prod_{i=l+1}^{n+1} D_{p_{n+1}(i)} = \prod_{i=l+1}^{n+1} D_{p_{n+1}(i)} \cdot D_{n+1}. \tag{1.25}$$

From (1.24) and (1.25), we find that

$$\prod_{i=1}^{n+1} D_{p_{n+1}(i)} = \prod_{i=1}^{l-1} D_{p_{n+1}(i)} \cdot \prod_{i=l+1}^{n+1} D_{p_{n+1}(i)} \cdot D_{n+1}. \tag{1.26}$$

Since $p_{n+1} : \{1, 2, \ldots n, n+1\} \to \{1, 2, \ldots n, n+1\}$ is a one-to-one function and since $p_{n+1}(l) = n+1$, we obtain that $p_{n+1}(i) \neq n+1$ for all $1 \le i \neq l \le n+1$. In other words,

$$1 \le p_{n+1}(i) \le n, \quad \forall\ 1 \le i \neq l \le n+1.$$

Thus, the function $p_n : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$ given by

$$p_n(i) = \begin{cases} p_{n+1}(i) & \text{if} \quad 1 \le i \le l-1; \\ p_{n+1}(i+1) & \text{if} \quad l+1 \le i \le n+1, \end{cases} \tag{1.27}$$

is well defined and one-to-one.

From the definition (1.27), it follows that

$$\prod_{i=1}^{l-1} D_{p_{n+1}(i)} \cdot \prod_{i=l+1}^{n+1} D_{p_{n+1}(i)} = \prod_{i=1}^{l-1} D_{p_{n+1}(i)} \cdot \prod_{i=l}^{n} D_{p_{n+1}(i+1}$$

$$= \prod_{i=1}^{l-1} D_{p_n(i)} \cdot \prod_{i=l}^{n} D_{p_n(i)}$$

$$= \prod_{i=1}^{n} D_{p_n(i)}, \qquad (1.28)$$

and therefore, from (1.26) and (1.28), we find that

$$\prod_{i=1}^{n+1} D_{p_{n+1}(i)} = \prod_{i=1}^{n} D_{p_n(i)} \cdot D_{n+1}. \qquad (1.29)$$

Recall from the induction hypothesis (1.22) that

$$\prod_{i=1}^{n} D_{p_n(i)} = \prod_{i=1}^{n} D_i. \qquad (1.30)$$

Then, from (1.29) and (1.30), we obtain that

$$\prod_{i=1}^{n+1} D_{p_{n+1}(i)} = \prod_{i=1}^{n} D_i \cdot D_{n+1} = \prod_{i=1}^{n+1} D_i,$$

which is what we wanted to show; see (1.23).

We conclude that (1.21) is proved by induction. □

Problem 6: (i) Let A be an $n \times n$ matrix and let L be an $n \times n$ nonsingular lower triangular matrix. Show that, if LA is a lower triangular matrix, then A is lower triangular. Show that, if AL is a lower triangular matrix, then A is lower triangular.

(ii) Let A be an $n \times n$ matrix and let U be an $n \times n$ nonsingular upper triangular matrix. Show that, if UA is an upper triangular matrix, then A is upper triangular. Show that, if AU is an upper triangular matrix, then A is upper triangular.

Solution: (i) Recall that the inverse of a nonsingular lower triangular matrix is a lower triangular matrix, and that the product of two lower triangular matrices is a lower triangular matrix.

Let $L_1 = LA$ be a lower triangular matrix. Since L is nonsingular, we obtain that

$$L^{-1}L_1 = L^{-1}(LA) = (L^{-1}L)A = I \cdot A = A,$$

since $LL^{-1} = I$.

Note that L^{-1} is a lower triangular matrix since it is the inverse of the lower triangular matrix L. Thus, $A = L^{-1}L_1$ is a lower triangular matrix since it is the product of the lower triangular matrices L^{-1} and L_1.

Similarly, let $L_2 = AL$ be a lower triangular matrix. Since L is nonsingular, we obtain that

$$L_2L^{-1} = (AL)L^{-1} = A(LL^{-1}) = A \cdot I = A.$$

Thus, $A = L_2L^{-1}$ is a lower triangular matrix since it is the product of the lower triangular matrices L_2 and L^{-1}.

(ii) Proofs along the lines of the proofs given at (i) for lower triangular matrices can be given to the corresponding results for upper triangular matrices; for completeness, we include them below.

However, if the results from (i) for lower triangular matrices are known, then the corresponding results for upper triangular matrices can be obtained by using matrix transposition as follows:

Recall that the inverse of a nonsingular upper triangular matrix is an upper triangular matrix, and that the product of two upper triangular matrices is an upper triangular matrix.

Let U be a nonsingular upper triangular matrix. Then, the transpose U^t of the matrix U is lower triangular and nonsingular since $\det(U^t) = \det(U) \neq 0$. Assume that the matrix UA is an upper triangular matrix. Then, $(UA)^t = A^tU^t$ is a lower triangular matrix.

Recall from (i) that, if L is a nonsingular lower triangular matrix and if AL is a lower triangular matrix, then A is lower triangular. In our case, U^t is a nonsingular lower triangular matrix and A^tU^t is lower triangular. We conclude that A^t is a lower triangular matrix, and therefore that A is an upper triangular matrix, which is what we wanted to show.

Similarly, let U be a nonsingular upper triangular matrix, and assume that the matrix AU is an upper triangular matrix. Then, $(AU)^t = U^tA^t$ is a lower triangular matrix.

Recall from (i) that, if L is a nonsingular lower triangular matrix and if LA is a lower triangular matrix, then A is lower triangular. In our case, U^t is a nonsingular lower triangular matrix and U^tA^t is lower triangular. We conclude that A^t is a lower triangular matrix, and therefore that A is an upper triangular matrix.

Proofs not using the results for lower triangular matrices are included below:

Let $U_1 = UA$ be an upper triangular matrix, where U is a nonsingular upper triangular matrix. Then,

$$U^{-1}U_1 \;=\; U^{-1}(UA) \;=\; (U^{-1}U)A \;=\; A,$$

since $U^{-1}U = I$.

Note that U^{-1} is an upper triangular matrix since it is the inverse of the upper triangular matrix U. Thus, $A = U^{-1}U_1$ is an upper triangular matrix since it is the product of the upper triangular matrices U^{-1} and U_1.

Similarly, let $U_2 = AU$ be an upper triangular matrix. Since U is nonsingular, we obtain that

$$U_2U^{-1} \;=\; (AU)U^{-1} \;=\; A(UU^{-1}) \;=\; A.$$

Thus, $A = U_2U^{-1}$ is an upper triangular matrix since it is the product of the upper triangular matrices U_2 and U^{-1}. □

Problem 7: Let A be a nonsingular matrix, and let k be a positive integer. Define A^{-k} as the k–th power of the inverse matrix of A, i.e., let $A^{-k} = \left(A^{-1}\right)^k$. Show that this definition is consistent, i.e., show that $A^k \cdot A^{-k} = A^{-k} \cdot A^k = I$.

Solution: Since $A^{-k} = \left(A^{-1}\right)^k$, we obtain that

$$\begin{aligned} A^k \cdot A^{-k} &= A^k \cdot \left(A^{-1}\right)^k \\ &= A^{k-1} \cdot A \cdot A^{-1} \cdot \left(A^{-1}\right)^{k-1} = A^{k-1} \cdot \left(A^{-1}\right)^{k-1} \\ &= A^{k-2} \cdot A \cdot A^{-1} \cdot \left(A^{-1}\right)^{k-2} = A^{k-2} \cdot \left(A^{-1}\right)^{k-2} \\ &= \ldots \\ &= A^2 \cdot \left(A^{-1}\right)^2 \\ &= A \cdot A \cdot A^{-1} \cdot A^{-1} = A \cdot A^{-1} \\ &= I; \end{aligned}$$

here we used repeatedly the fact that $A \cdot A^{-1} = I$.

Thus, $A^k \cdot A^{-k} = I$. We conclude that A^{-k} is the inverse matrix of A^k, and therefore $A^{-k} \cdot A^k = I$. □

Problem 8: (i) Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{pmatrix}.$$

Compute M^2, M^3, M^4.

(ii) Let

$$C = I + M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{pmatrix}.$$

Compute C^m, where $m \geq 2$ is a positive integer.

Solution: (i) By direct computation, we find that

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \end{pmatrix}; \quad M^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{pmatrix}; \quad M^4 = 0. \tag{1.31}$$

(ii) Recall that, if A and B are square matrices of the same size such that $AB = BA$, then the following version of the binomial formula holds true:

$$(A+B)^m = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} A^j B^{m-j}, \tag{1.32}$$

where m is a positive integer and the binomial coefficient $\begin{pmatrix} m \\ j \end{pmatrix}$ is given by

$$\begin{pmatrix} m \\ j \end{pmatrix} = \frac{m!}{j!\,(m-j)!}, \tag{1.33}$$

where $k! = 1 \cdot 2 \cdot \ldots \cdot k$. Also, by definition, $A^0 = B^0 = I$.

Let $A = M$ and $B = I$ in (1.32), and note from (1.31) that $M^j = 0$ for all $j \geq 4$. Then,

$$\begin{aligned} C^m &= (M+I)^m = \sum_{j=0}^{m} \binom{m}{j} M^j = \sum_{j=0}^{3} \binom{m}{j} M^j \\ &= \binom{m}{0} I + \binom{m}{1} M + \binom{m}{2} M^2 + \binom{m}{3} M^3. \end{aligned} \tag{1.34}$$

From (1.33), we obtain that

$$\binom{m}{0} = 1; \ \binom{m}{1} = m; \ \binom{m}{2} = \frac{m(m-1)}{2}; \ \binom{m}{3} = \frac{m(m-1)(m-2)}{6}.$$

Then, using the values of M^2 and M^3 from (1.31), we conclude from (1.34) that

$$\begin{aligned} C^m &= \binom{m}{0} I + \binom{m}{1} M + \binom{m}{2} M^2 + \binom{m}{3} M^3 \\ &= I + mM + \frac{m(m-1)}{2} M^2 + \frac{m(m-1)(m-2)}{6} M^3 \\ &= I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3m & 0 & 0 & 0 \\ m & -m & 0 & 0 \\ -m & 2m & m & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{3m(m-1)}{2} & 0 & 0 & 0 \\ \frac{7m(m-1)}{2} & -\frac{m(m-1)}{2} & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{3m(m-1)(m-2)}{6} & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3m & 1 & 0 & 0 \\ \frac{5m-3m^2}{2} & -m & 1 & 0 \\ \frac{-33m+30m^2-3m^3}{6} & \frac{5m-m^2}{2} & m & 1 \end{pmatrix} \quad \square \end{aligned}$$

Problem 9: Let L be an $n \times n$ lower triangular matrix with entries equal to 0 on the main diagonal, i.e., with $L(i,i) = 0$ for $i = 1 : n$.

(i) Show that $L^n = 0$;

(ii) Compute $(I+L)^m$ in terms of $L, L^2, \ldots, L^{n-1}$, where $m \geq n$ is a positive integer.

Solution: (i) To show that $L^n = 0$, we will prove by induction the following result:

"The first i rows of the matrix L^i are zero vectors, for any $1 \leq i \leq n$."

For $i = 1$, the first row of the matrix L is a zero vector, since L is lower triangular and therefore $L(1,k) = 0$ for $2 \leq k \leq n$, and $L(1,1) = 0$ since the main diagonal entries of L are 0.

Let $2 \leq i \leq n-1$. Assume that the first i rows of the matrix L^i are zero vectors and let

$$L^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r_{i+1}^{(i)} \\ \vdots \\ r_n^{(i)} \end{pmatrix} \tag{1.35}$$

be the row form of L^i. We will show that the first $(i+1)$ rows of the matrix L^{i+1} are zero vectors.

Let

$$L^{i+1} = \begin{pmatrix} r_1^{(i+1)} \\ r_2^{(i+1)} \\ \vdots \\ r_n^{(i+1)} \end{pmatrix}$$

be the row form of the matrix L^{i+1}. Note that $L^{i+1} = L \cdot L^i$ and therefore the j–th row of L^{i+1} is a linear combination of the rows of L^i with coefficients the entries of the j–th row of L, i.e.,

$$r_j^{(i+1)} = \sum_{k=1}^{n} L(j,k)\; r_k^{(i)}, \quad \forall\; j = 1:n. \tag{1.36}$$

Note that $L(j,j) = 0$, since all the main diagonal entries of L are 0, and $L(j,k) = 0$ for $j+1 \leq k \leq n$, since L is a lower triangular matrix. Then, (1.36) can be written as

$$r_j^{(i+1)} = \sum_{k=1}^{j-1} L(j,k)\; r_k^{(i)}, \quad \forall\; j = 1:n. \tag{1.37}$$

Since $r_1^{(i)} = r_2^{(i)} = \ldots = r_i^{(i)}$, see (1.35), it follows from (1.37) that

$$\begin{aligned}
r_1^{(i+1)} &= 0; \\
r_2^{(i+1)} &= L(2,1)\; r_1^{(i)} = 0; \\
r_3^{(i+1)} &= \sum_{k=1}^{2} L(3,k)\; r_k^{(i)} \;=\; L(3,1)\; r_1^{(i)} + L(3,2)\; r_2^{(i)} \;=\; 0; \\
&\;\;\vdots \\
r_{i+1}^{(i+1)} &= \sum_{k=1}^{i} L(i+1,k)\; r_k^{(i)} \\
&= L(i+1,1)\; r_1^{(i)} + L(i+1,2)\; r_2^{(i)} + \ldots + L(i+1,i)\; r_i^{(i)} \\
&= 0.
\end{aligned}$$

In other words, $r_1^{(i+1)} = r_2^{(i+1)} = \ldots = r_{i+1}^{(i+1)} = 0$, i.e., the first $(i+1)$ rows of the matrix L^{i+1} are zero vectors.

This completes the proof by induction of the fact that the first i rows of the matrix L^i are zero vectors, for all $1 \leq i \leq n$.

Thus, the first n rows of the matrix L^n are zero vectors, and, since L^n is an $n \times n$ matrix, we conclude that $L^n = 0$.

(ii) Let $m \geq n$. Let $A = L$ and $B = I$ in the binomial formula (1.32). Since $L^j = 0$ for all $j \geq n$, we obtain that

$$\begin{aligned} (I+L)^m &= (L+I)^m = \sum_{j=0}^{m} \binom{m}{j} L^j = \sum_{j=0}^{n-1} \binom{m}{j} L^j \\ &= I + mL + \binom{m}{2} L^2 + \ldots + + \binom{m}{n-1} L^{n-1}. \quad \square \end{aligned}$$

Problem 10: Let A and B be square matrices of the same size with nonnegative entries and such that the sum of the entries in each row is equal to 1. Show that the matrix AB has the same properties, i.e., show that all the entries of the matrix AB are nonnegative and the sum of the entries in each row of AB is equal to 1.

Solution: A matrix with nonnegative entries such that the sum of the entries in each row is equal to 1 is called a probability matrix. Thus, the problem can be restated as follows:

Let A and B be probability matrices of the same size. Show that AB is a probability matrix.

We first establish the following equivalent definition for a probability matrix:

The $n \times n$ matrix M is a probability matrix if and only if all the entries of M are nonnegative and

$$M\mathbf{1} = \mathbf{1}, \tag{1.38}$$

where $\mathbf{1}$ is the $n \times 1$ column vector with all entries equal to 1.

To see this, let $M = \text{row}\,(r_j)_{j=1:n}$ be the row form of the matrix M, where r_j is an $1 \times n$ row vector, for $j = 1 : n$. The sum of all the entries in the j-th row r_j of M can be written as follows:[1]

$$\sum_{k=1}^{n} r_j(k) = r_j\mathbf{1}. \tag{1.39}$$

Thus, the definition of a probability matrix as a matrix with the sum of the entries in each row equal to 1 can be written as

$$\begin{aligned} & \sum_{k=1}^{n} r_j(k) = 1, \quad \forall\, j = 1 : n \\ \Longleftrightarrow\quad & r_j\mathbf{1} = 1, \quad \forall\, j = 1 : n \\ \Longleftrightarrow\quad & (r_j\mathbf{1})_{j=1:n} = \mathbf{1} \\ \Longleftrightarrow\quad & M\mathbf{1} = \mathbf{1}, \end{aligned}$$

[1] Note that r_j is an $1 \times n$ vector and $\mathbf{1}$ is an $n \times 1$ vector, and therefore the expression $r_j\mathbf{1}$ from (1.39) is consistent.

since $M\mathbf{1} = (r_j\mathbf{1})_{j=1:n}$ if $M = \text{row}\,(r_j)_{j=1:n}$ is the row form of M.

In other words, we established that (1.38) is an equivalent condition for M to be a probability matrix.

Let A and B be probability matrices. Then all the entries of A and B are nonnegative, and therefore all the entries of AB are also nonnegative.[2] From (1.38), it follows that

$$A\mathbf{1} = \mathbf{1} \quad \text{and} \quad B\mathbf{1} = \mathbf{1},$$

and therefore

$$(AB)\mathbf{1} \;=\; A(B\mathbf{1}) \;=\; A\mathbf{1} \;=\; \mathbf{1}. \tag{1.40}$$

Then, from (1.38) and (1.40), we conclude that AB is a probability matrix, which is what we wanted to show. □

Problem 11: The covariance matrix of five random variables is

$$\Sigma \;=\; \begin{pmatrix} 1 & -0.525 & 1.375 & -0.075 & -0.75 \\ -0.525 & 2.25 & 0.1875 & 0.1875 & -0.675 \\ 1.375 & 0.1875 & 6.25 & 0.4375 & -1.875 \\ -0.075 & 0.1875 & 0.4375 & 0.25 & 0.3 \\ -0.75 & -0.675 & -1.875 & 0.3 & 9 \end{pmatrix}. \tag{1.41}$$

Find the correlation matrix of these random variables.

Solution: Recall that the correlation matrix Ω and the covariance matrix Σ of n nonconstant random variables $X_1, X_2, \ldots X_n$ satisfy the following relationship:

$$\Omega \;=\; (D_\sigma)^{-1}\Sigma(D_\sigma)^{-1}, \tag{1.42}$$

where D_σ is the diagonal matrix given by $D_\sigma = \text{diag}(\sigma_i)_{i=1:n}$; here, σ_i denotes the standard deviation of the random variable X_i, for $i = 1 : n$.

From (1.41), we find that the standard deviations of the five random variables are

$$\sigma_1 = \sqrt{\Sigma(1,1)} = 1;\ \sigma_2 = \sqrt{\Sigma(2,2)} = 1.5;\ \sigma_3 = \sqrt{\Sigma(3,3)} = 2.5;$$

$$\sigma_4 = \sqrt{\Sigma(4,4)} = 0.5;\ \sigma_5 = \sqrt{\Sigma(5,5)} = 3,$$

and therefore

$$D_\sigma \;=\; \text{diag}(\sigma_i)_{i=1:5} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}. \tag{1.43}$$

[2] Formally, the fact that all the entries of the matrix AB are nonnegative if all the entries of A and B are nonnegative can be proved as follows:

$$(AB)(j,k) \;=\; \sum_{i=1}^{n} A(j,i)B(i,k) \;\geq\; 0, \quad \forall\ 1 \leq j,k \leq n,$$

if $A(j,i) \geq 0$ for all $1 \leq i,j \leq n$ and $B(i,k) \geq 0$ for all $1 \leq i,k \leq n$.

Then, from (1.42) and (1.43), we obtain that the correlation matrix of the five random variables is

$$\Omega = \begin{pmatrix} 1 & -0.35 & 0.55 & -0.15 & -0.25 \\ -0.35 & 1 & 0.05 & 0.25 & -0.15 \\ 0.55 & 0.05 & 1 & 0.35 & -0.25 \\ -0.15 & 0.25 & 0.35 & 1 & 0.20 \\ -0.25 & -0.15 & -0.25 & 0.20 & 1 \end{pmatrix}. \quad \square$$

Problem 12: The correlation matrix of five random variables is

$$\Omega = \begin{pmatrix} 1 & -0.25 & 0.15 & -0.05 & -0.30 \\ -0.25 & 1 & -0.10 & -0.25 & 0.10 \\ 0.15 & -0.10 & 1 & 0.20 & 0.05 \\ -0.05 & -0.25 & 0.20 & 1 & 0.10 \\ -0.30 & 0.10 & 0.05 & 0.10 & 1 \end{pmatrix} \tag{1.44}$$

(i) Compute the covariance matrix of these random variables if their standard deviations are 0.25, 0.5, 1, 2, and 4, in this order.

(ii) Compute the covariance matrix of these random variables if their standard deviations are 4, 2, 1, 0.5, and 0.25, in this order.

Solution: The covariance matrix Σ and the correlation matrix Ω of n random variables $X_1, X_2, \ldots X_n$ satisfy the following relationship:

$$\Sigma = D_\sigma \Omega D_\sigma, \tag{1.45}$$

where D_σ is the diagonal matrix given by $D_\sigma = \mathrm{diag}(\sigma_i)_{i=1:n}$; here, σ_i denotes the standard deviation of the random variable X_i, for $i = 1 : n$.

(i) If the standard deviations of the random variables are 0.25, 0.5, 1, 2, and 4, then

$$D_\sigma = \begin{pmatrix} 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \tag{1.46}$$

and, from (1.45) and using (1.44) and (1.46), we obtain that the covariance matrix of the random variables is

$$\Sigma = \begin{pmatrix} 0.0625 & -0.0312 & 0.0375 & -0.025 & -0.3 \\ -0.0312 & 0.25 & -0.05 & -0.25 & 0.2 \\ 0.0375 & -0.05 & 1 & 0.4 & 0.2 \\ -0.025 & -0.25 & 0.4 & 4 & 0.8 \\ -0.3 & 0.20 & 0.2 & 0.8 & 16 \end{pmatrix}.$$

(ii) If the standard deviations of the random variables are 4, 2, 1, 0.5, and 0.25, then

$$D_\sigma = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.25 \end{pmatrix}, \tag{1.47}$$

and, from (1.45) and using (1.44) and (1.47), we obtain that the covariance matrix of the random variables is

$$\Sigma = \begin{pmatrix} 16 & -2 & 0.6 & -0.1 & -0.3 \\ -2 & 4 & -0.2 & -0.25 & 0.05 \\ 0.6 & -0.2 & 1 & 0.1 & 0.0125 \\ -0.1 & -0.25 & 0.1 & 0.25 & 0.0125 \\ -0.3 & 0.05 & 0.0125 & 0.0125 & 0.0625 \end{pmatrix}. \quad \Box$$

Problem 13: The file *indeces-jul26-aug9-2012.xlsx* from fepress.org/nla-primer contains the July 26, 2012 – August 9, 2012 end of day values of Dow Jones, Nasdaq, and S&P 500.

(i) Compute the daily percentage returns of the three indices over the given time period.

(ii) Compute the covariance matrix of the daily percentage returns of the three indices.

(iii) Compute the daily log returns of the three indices over the given time period.

(iv) Compute the covariance matrix of the daily log returns of the three indices.

Solution: The July 26, 2012 – August 9, 2012 end of day values of Dow Jones, Nasdaq, and S&P 500 were:

Date	Dow Jones	NASDAQ	S&P 500
07/26/2012	12887.93	2893.25	1360.02
07/27/2012	13075.66	2958.09	1385.97
07/30/2012	13073.01	2945.84	1385.30
07/31/2012	13008.68	2939.52	1379.32
08/01/2012	12976.13	2920.21	1375.32
08/02/2012	12878.88	2909.77	1365.00
08/03/2012	13096.17	2967.90	1390.99
08/06/2012	13117.51	2989.91	1394.23
08/07/2012	13168.60	3015.86	1401.35
08/08/2012	13175.64	3011.25	1402.22
08/09/2012	13165.19	3018.64	1402.80

(i) The percentage return between times t_1 and t_2 of an asset with price $S(t)$ at time t is given by $\frac{S(t_2)-S(t_1)}{S(t_1)}$. Then, the time series matrix of the daily percentage returns of the three indices between 07/26/2012 and 08/09/2012 is

$$T_{\mathbf{x}}^{(p)} = \begin{pmatrix} 0.014566 & 0.022411 & 0.019081 \\ -0.000203 & -0.004141 & -0.000483 \\ -0.004921 & -0.002145 & -0.004317 \\ -0.002502 & -0.006569 & -0.002900 \\ -0.007495 & -0.003575 & -0.007504 \\ 0.016872 & 0.019978 & 0.019040 \\ 0.001629 & 0.007416 & 0.002329 \\ 0.003895 & 0.008679 & 0.005107 \\ 0.000535 & -0.001529 & 0.000621 \\ -0.000793 & 0.002454 & 0.000414 \end{pmatrix}, \qquad (1.48)$$

where, e.g., the percentage return on 08/08/2012 of NASDAQ is

$$\frac{3011.25 - 3015.86}{3015.86} = -0.001529 = T_{\mathbf{x}}^{(p)}(9,2).$$

(ii) The sample means of the daily returns of the three indices are 0.002158 (Dow Jones), 0.004298 (NASDAQ), and 0.003139 (S&P 500). By subtracting the sample mean of each column from $T_{\mathbf{x}}^{(p)}$, see (1.48), we obtain that the time series matrix of mean normalized daily percentage returns of the three indices is

$$\overline{T}_{\mathbf{x}}^{(p)} = \begin{pmatrix} 0.012408 & 0.018113 & 0.015942 \\ -0.002361 & -0.008439 & -0.003622 \\ -0.007079 & -0.006443 & -0.007456 \\ -0.004661 & -0.010867 & -0.006039 \\ -0.009653 & -0.007873 & -0.010642 \\ 0.014713 & 0.015680 & 0.015902 \\ -0.000529 & 0.003118 & -0.000809 \\ 0.001736 & 0.004381 & 0.001968 \\ -0.001624 & -0.005826 & -0.002518 \\ -0.002952 & -0.001844 & -0.002725 \end{pmatrix}.$$

Note that $\overline{T}_{\mathbf{x}}^{(p)}$ is a 10×3 matrix, corresponding to $N = 10$ daily returns computed from the 11 daily closes.

The covariance matrix of the daily percentage returns of the three indices is

$$\begin{aligned} \widehat{\Sigma}_{\mathbf{x}}^{(p)} &= \frac{1}{N-1} (\overline{T}_{\mathbf{x}}^{(p)})^t \overline{T}_{\mathbf{x}}^{(p)} \\ &= \begin{pmatrix} 0.000061742 & 0.000074276 & 0.000071106 \\ 0.000074276 & 0.000103667 & 0.000087988 \\ 0.000071106 & 0.000087988 & 0.000082637 \end{pmatrix}. \end{aligned}$$

(iii) The log return between times t_1 and t_2 of an asset with price $S(t)$ at time t is given by $\ln\left(\frac{S(t_2)}{S(t_1)}\right)$. Then, the time series matrix of the daily log returns of the three indices between 07/26/2012 and 08/09/2012 is

$$T_{\mathbf{x}}^{(log)} = \begin{pmatrix} 0.014461 & 0.022163 & 0.018901 \\ -0.000203 & -0.004150 & -0.000484 \\ -0.004933 & -0.002148 & -0.004326 \\ -0.002505 & -0.006591 & -0.002904 \\ -0.007523 & -0.003581 & -0.007532 \\ 0.016731 & 0.019781 & 0.018861 \\ 0.001628 & 0.007389 & 0.002327 \\ 0.003887 & 0.008642 & 0.005094 \\ 0.000534 & -0.001530 & 0.000621 \\ -0.000793 & 0.002451 & 0.000414 \end{pmatrix}, \qquad (1.49)$$

where, e.g., the log return on 08/01/2012 of S&P 500 is

$$\ln\left(\frac{1375.32}{1379.32}\right) = -0.002904 = T_{\mathbf{x}}^{(log)}(4,3).$$

(iv) The sample means of the log returns of the three indices are 0.002129 (Dow Jones), 0.004243 (NASDAQ), and 0.003097 (S&P 500). By subtracting the sample mean of each column from $T_{\mathbf{x}}^{(log)}$, see (1.49), we obtain that the time series matrix of mean normalized log percentage returns of the three indices is

$$\overline{T}_{\mathbf{x}}^{(log)} = \begin{pmatrix} 0.012333 & 0.017921 & 0.015804 \\ -0.002331 & -0.008392 & -0.003581 \\ -0.007061 & -0.006390 & -0.007423 \\ -0.004634 & -0.010833 & -0.006001 \\ -0.009651 & -0.007824 & -0.010629 \\ 0.014603 & 0.015538 & 0.015764 \\ -0.000500 & 0.003146 & -0.000771 \\ 0.001759 & 0.004399 & 0.001997 \\ -0.001594 & -0.005772 & -0.002476 \\ -0.002922 & -0.001791 & -0.002684 \end{pmatrix}.$$

Note that $\overline{T}_{\mathbf{x}}^{(log)}$ is a 10×3 matrix, corresponding to $N = 10$ daily returns computed from the 11 daily closes.

The covariance matrix of the daily log returns of the three indices is

$$\begin{aligned} \widehat{\Sigma}_{\mathbf{x}}^{(log)} &= \frac{1}{N-1} (\overline{T}_{\mathbf{x}}^{(log)})^t \overline{T}_{\mathbf{x}}^{(log)} \\ &= \begin{pmatrix} 0.000061075 & 0.000073212 & 0.000070216 \\ 0.000073212 & 0.000102023 & 0.000086587 \\ 0.000070216 & 0.000086587 & 0.000081456 \end{pmatrix}. \quad \square \end{aligned}$$

Problem 14: The file *indices-july2011.xlsx* from fepress.org/nla-primer contains the January 2011 – July 2011 end of day values of nine major US indices.

(i) Compute the sample covariance matrix of the daily percentage returns of the indices, and the corresponding sample correlation matrix.

Compute the sample covariance and correlation matrices for daily log returns, and compare them with the corresponding matrices for daily percentage returns.

(ii) Compute the sample covariance matrix of the weekly percentage returns of the indices, and the corresponding sample correlation matrix.

Compute the sample covariance and correlation matrices for weekly log returns, and compare them with the corresponding matrices for weekly percentage returns.

(iii) Compute the sample covariance matrix of the monthly percentage returns of the indices, and the corresponding sample correlation matrix.

Compute the sample covariance and correlation matrices for monthly log returns, and compare them with the corresponding matrices for monthly percentage returns.

(iv) Comment on the differences between the sample covariance and correlation matrices for daily, weekly, and monthly returns.

Solution: (i) The sample covariance matrix of the daily percentage returns of the indices, and the corresponding sample correlation matrix are

$$10^{-6} \cdot \begin{pmatrix} 100.04 & 67.04 & 96.67 & 40.04 & 82.77 & 78.90 & 74.35 & 71.11 & 54.56 \\ 67.04 & 58.16 & 71.29 & 38.01 & 65.29 & 61.38 & 59.63 & 52.87 & 41.99 \\ 96.67 & 71.29 & 135.88 & 45.36 & 86.15 & 83.05 & 77.46 & 60.15 & 57.46 \\ 40.04 & 38.01 & 45.36 & 44.84 & 43.20 & 40.10 & 39.37 & 34.60 & 30.43 \\ 82.77 & 65.29 & 86.15 & 43.20 & 82.37 & 74.46 & 71.37 & 73.05 & 54.01 \\ 78.90 & 61.38 & 83.05 & 40.10 & 74.46 & 69.77 & 66.94 & 61.80 & 46.55 \\ 74.35 & 59.63 & 77.46 & 39.36 & 71.37 & 66.94 & 65.17 & 58.66 & 43.66 \\ 71.11 & 52.87 & 60.15 & 34.60 & 73.05 & 61.80 & 58.66 & 103.96 & 50.01 \\ 54.56 & 41.99 & 57.46 & 30.43 & 54.01 & 46.55 & 43.66 & 50.01 & 110.84 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.8790 & 0.8292 & 0.5979 & 0.9118 & 0.9444 & 0.9208 & 0.6972 & 0.5181 \\ 0.8790 & 1 & 0.8019 & 0.7444 & 0.9433 & 0.9634 & 0.9686 & 0.6799 & 0.5229 \\ 0.8292 & 0.8019 & 1 & 0.5812 & 0.8143 & 0.8529 & 0.8232 & 0.5061 & 0.4682 \\ 0.5979 & 0.7444 & 0.5812 & 1 & 0.7109 & 0.7169 & 0.7282 & 0.5067 & 0.4316 \\ 0.9118 & 0.9433 & 0.8143 & 0.7109 & 1 & 0.9821 & 0.9742 & 0.7894 & 0.5652 \\ 0.9444 & 0.9634 & 0.8529 & 0.7169 & 0.9821 & 1 & 0.9927 & 0.7256 & 0.5293 \\ 0.9208 & 0.9686 & 0.8232 & 0.7282 & 0.9742 & 0.9927 & 1 & 0.7126 & 0.5137 \\ 0.6972 & 0.6799 & 0.5061 & 0.5067 & 0.7894 & 0.7256 & 0.7126 & 1 & 0.4659 \\ 0.5181 & 0.5229 & 0.4682 & 0.4316 & 0.5652 & 0.5293 & 0.5137 & 0.4659 & 1 \end{pmatrix}$$

The sample covariance matrix of the daily log returns of the indices, and the corresponding sample correlation matrix are

$$10^{-6} \cdot \begin{pmatrix} 100.40 & 67.27 & 97.25 & 40.15 & 83.13 & 79.21 & 74.63 & 71.38 & 54.60 \\ 67.27 & 58.32 & 71.57 & 38.09 & 65.53 & 61.58 & 59.82 & 53.06 & 41.99 \\ 97.25 & 71.57 & 136.67 & 45.46 & 86.59 & 83.46 & 77.81 & 60.39 & 57.57 \\ 40.15 & 38.09 & 45.46 & 44.90 & 43.34 & 40.20 & 39.47 & 34.76 & 30.56 \\ 83.13 & 65.53 & 86.59 & 43.34 & 82.72 & 74.76 & 71.65 & 73.33 & 54.04 \\ 79.21 & 61.58 & 83.46 & 40.20 & 74.76 & 70.03 & 67.18 & 62.04 & 46.58 \\ 74.63 & 59.82 & 77.81 & 39.47 & 71.65 & 67.18 & 65.39 & 58.88 & 43.66 \\ 71.38 & 53.06 & 60.39 & 34.76 & 73.33 & 62.04 & 58.88 & 104.16 & 50.06 \\ 54.60 & 41.99 & 57.57 & 30.56 & 54.04 & 46.58 & 43.66 & 50.06 & 111.05 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.8791 & 0.8302 & 0.5980 & 0.9121 & 0.9447 & 0.9210 & 0.6981 & 0.5171 \\ 0.8791 & 1 & 0.8017 & 0.7444 & 0.9435 & 0.9635 & 0.9687 & 0.6808 & 0.5218 \\ 0.8302 & 0.8017 & 1 & 0.5803 & 0.8144 & 0.8531 & 0.8232 & 0.5062 & 0.4673 \\ 0.5980 & 0.7444 & 0.5803 & 1 & 0.7111 & 0.7169 & 0.7283 & 0.5083 & 0.4327 \\ 0.9121 & 0.9435 & 0.8144 & 0.7111 & 1 & 0.9822 & 0.9742 & 0.7900 & 0.5638 \\ 0.9447 & 0.9635 & 0.8531 & 0.7169 & 0.9822 & 1 & 0.9927 & 0.7264 & 0.5281 \\ 0.9210 & 0.9687 & 0.8232 & 0.7283 & 0.9742 & 0.9927 & 1 & 0.7134 & 0.5124 \\ 0.6981 & 0.6808 & 0.5062 & 0.5083 & 0.7900 & 0.7264 & 0.7134 & 1 & 0.4654 \\ 0.5171 & 0.5218 & 0.4673 & 0.4327 & 0.5638 & 0.5281 & 0.5124 & 0.4654 & 1 \end{pmatrix}$$

Since log returns are excellent approximations for percentage returns in the case of small returns such as daily returns, the sample covariance and correlation matrices for daily percentage returns and for daily log returns are very close to each other.

(ii) The sample covariance matrix of the weekly percentage returns[3] of the indices,

[3] The weekly returns are computed using the end of day price from the last trading day of the prior week and the end of day price from the last trading day of the current week, i.e., the "Friday to Friday" convention.

and the corresponding sample correlation matrix are

$$10^{-6} \cdot \begin{pmatrix} 499.83 & 344.14 & 453.90 & 208.79 & 398.01 & 391.49 & 378.60 & 411.23 & 317.36 \\ 344.14 & 289.74 & 355.05 & 181.11 & 306.84 & 302.12 & 297.39 & 276.41 & 244.01 \\ 453.90 & 355.05 & 678.75 & 253.15 & 383.24 & 392.69 & 371.69 & 276.45 & 306.13 \\ 208.79 & 181.11 & 253.15 & 222.43 & 179.49 & 192.78 & 190.84 & 104.43 & 188.94 \\ 398.01 & 306.84 & 383.24 & 179.49 & 372.05 & 344.68 & 336.28 & 391.27 & 271.59 \\ 391.49 & 302.12 & 392.69 & 192.78 & 344.68 & 335.21 & 326.99 & 327.86 & 263.23 \\ 378.60 & 297.39 & 371.69 & 190.84 & 336.28 & 326.99 & 323.48 & 316.18 & 256.98 \\ 411.23 & 276.41 & 276.45 & 104.43 & 391.27 & 327.86 & 316.18 & 656.36 & 258.95 \\ 317.36 & 244.01 & 306.13 & 188.94 & 271.59 & 263.23 & 256.98 & 258.95 & 410.64 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.9043 & 0.7793 & 0.6262 & 0.9230 & 0.9564 & 0.9416 & 0.7180 & 0.7005 \\ 0.9043 & 1 & 0.8006 & 0.7134 & 0.9346 & 0.9694 & 0.9714 & 0.6338 & 0.7074 \\ 0.7793 & 0.8006 & 1 & 0.6515 & 0.7626 & 0.8233 & 0.7932 & 0.4142 & 0.5799 \\ 0.6262 & 0.7134 & 0.6515 & 1 & 0.6240 & 0.7060 & 0.7115 & 0.2733 & 0.6252 \\ 0.9230 & 0.9346 & 0.7626 & 0.6240 & 1 & 0.9760 & 0.9694 & 0.7918 & 0.6948 \\ 0.9564 & 0.9694 & 0.8233 & 0.7060 & 0.9760 & 1 & 0.9930 & 0.6990 & 0.7095 \\ 0.9416 & 0.9714 & 0.7932 & 0.7115 & 0.9694 & 0.9930 & 1 & 0.6862 & 0.7051 \\ 0.7180 & 0.6338 & 0.4142 & 0.2733 & 0.7918 & 0.6990 & 0.6862 & 1 & 0.4988 \\ 0.7005 & 0.7074 & 0.5799 & 0.6252 & 0.6948 & 0.7095 & 0.7051 & 0.4988 & 1 \end{pmatrix}$$

The sample covariance matrix of the weekly log returns of the indices, and the corresponding sample correlation matrix are

$$10^{-6} \cdot \begin{pmatrix} 493.00 & 337.88 & 448.96 & 206.47 & 391.99 & 385.37 & 372.19 & 407.91 & 310.63 \\ 337.88 & 284.42 & 349.79 & 178.75 & 301.59 & 296.86 & 291.97 & 272.65 & 237.68 \\ 448.96 & 349.79 & 676.39 & 249.89 & 378.35 & 388.15 & 366.51 & 272.85 & 298.99 \\ 206.47 & 178.75 & 249.89 & 223.35 & 177.08 & 190.72 & 188.71 & 102.76 & 185.58 \\ 391.99 & 301.59 & 378.35 & 177.08 & 366.86 & 339.41 & 330.83 & 388.38 & 265.44 \\ 385.37 & 296.86 & 388.15 & 190.72 & 339.41 & 329.97 & 321.52 & 324.24 & 257.08 \\ 372.19 & 291.97 & 366.51 & 188.71 & 330.83 & 321.52 & 317.81 & 312.58 & 250.61 \\ 407.91 & 272.65 & 272.85 & 102.76 & 388.38 & 324.24 & 312.58 & 655.52 & 256.49 \\ 310.63 & 237.68 & 298.99 & 185.58 & 265.44 & 257.08 & 250.61 & 256.49 & 406.67 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.9023 & 0.7775 & 0.6222 & 0.9217 & 0.9555 & 0.9403 & 0.7175 & 0.6937 \\ 0.9023 & 1 & 0.7975 & 0.7092 & 0.9337 & 0.9690 & 0.9711 & 0.6314 & 0.6989 \\ 0.7775 & 0.7975 & 1 & 0.6429 & 0.7595 & 0.8216 & 0.7905 & 0.4098 & 0.5701 \\ 0.6222 & 0.7092 & 0.6429 & 1 & 0.6186 & 0.7025 & 0.7083 & 0.2686 & 0.6158 \\ 0.9217 & 0.9337 & 0.7595 & 0.6186 & 1 & 0.9755 & 0.9689 & 0.7920 & 0.6872 \\ 0.9555 & 0.9690 & 0.8216 & 0.7025 & 0.9755 & 1 & 0.9929 & 0.6972 & 0.7018 \\ 0.9403 & 0.9711 & 0.7905 & 0.7083 & 0.9689 & 0.9929 & 1 & 0.6848 & 0.6971 \\ 0.7175 & 0.6314 & 0.4098 & 0.2686 & 0.7920 & 0.6972 & 0.6848 & 1 & 0.4968 \\ 0.6937 & 0.6989 & 0.5701 & 0.6158 & 0.6872 & 0.7018 & 0.6971 & 0.4968 & 1 \end{pmatrix}$$

Weekly log returns are very good approximations for weekly percentage returns, and therefore the sample covariance and correlation matrices for weekly percentage returns and for weekly log returns are close to each other.

(iii) The monthly[4] percentage returns of the nine indices are

0.0178	0.0272	−0.0160	0.0108	0.0220	0.0226	0.0236	−0.0165	−0.0252
0.0304	0.0281	0.0119	0.0153	0.0368	0.0320	0.0289	0.0963	−0.0162
−0.0004	0.0076	0.0423	−0.0061	−0.0040	−0.0010	−0.0054	−0.0058	0.0055
0.0332	0.0398	0.0406	0.0387	0.0317	0.0285	0.0263	0.0488	0.0246
−0.0133	−0.0188	−0.0082	0.0170	−0.0224	−0.0135	−0.0181	−0.0185	−0.0099
−0.0218	−0.0124	−0.0084	−0.0066	−0.0187	−0.0183	−0.0168	−0.0383	−0.0314

[4] The monthly returns are computed using the end of day price from the last trading day of the prior month and the end of day price from the last trading day of the current month.

The sample covariance matrix of the monthly percentage returns of the indices and the corresponding sample correlation matrix are

$$10^{-6}\cdot\begin{pmatrix} 529.78 & 526.36 & 244.69 & 267.24 & 584.63 & 503.19 & 497.53 & 967.95 & 218.71 \\ 526.36 & 565.13 & 271.06 & 231.25 & 594.65 & 504.06 & 509.56 & 839.03 & 216.96 \\ 244.69 & 271.06 & 664.65 & 104.25 & 214.54 & 174.44 & 139.12 & 586.15 & 465.90 \\ 267.24 & 231.25 & 104.25 & 285.36 & 254.10 & 230.93 & 221.68 & 480.23 & 209.20 \\ 584.63 & 594.65 & 214.54 & 254.10 & 672.48 & 568.58 & 572.66 & 1076.07 & 162.90 \\ 503.19 & 504.06 & 174.44 & 230.93 & 568.58 & 488.24 & 488.07 & 899.60 & 151.06 \\ 497.53 & 509.56 & 139.12 & 221.68 & 572.66 & 488.07 & 494.77 & 852.96 & 119.97 \\ 967.95 & 839.03 & 586.15 & 480.23 & 1076.07 & 899.60 & 852.96 & 2609.89 & 397.89 \\ 218.71 & 216.96 & 465.90 & 209.20 & 162.90 & 151.06 & 119.97 & 397.89 & 431.37 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.9620 & 0.4124 & 0.6873 & 0.9795 & 0.9894 & 0.9718 & 0.8232 & 0.4575 \\ 0.9620 & 1 & 0.4423 & 0.5758 & 0.9646 & 0.9596 & 0.9637 & 0.6909 & 0.4394 \\ 0.4124 & 0.4423 & 1 & 0.2394 & 0.3209 & 0.3062 & 0.2426 & 0.4450 & 0.8701 \\ 0.6873 & 0.5758 & 0.2394 & 1 & 0.5801 & 0.6187 & 0.5900 & 0.5565 & 0.5963 \\ 0.9795 & 0.9646 & 0.3209 & 0.5801 & 1 & 0.9923 & 0.9928 & 0.8123 & 0.3025 \\ 0.9894 & 0.9596 & 0.3062 & 0.6187 & 0.9923 & 1 & 0.9930 & 0.7969 & 0.3292 \\ 0.9718 & 0.9637 & 0.2426 & 0.5900 & 0.9928 & 0.9930 & 1 & 0.7506 & 0.2597 \\ 0.8232 & 0.6909 & 0.4450 & 0.5565 & 0.8123 & 0.7969 & 0.7506 & 1 & 0.3750 \\ 0.4575 & 0.4394 & 0.8701 & 0.5963 & 0.3025 & 0.3292 & 0.2597 & 0.3750 & 1 \end{pmatrix}$$

The sample covariance matrix of the monthly log returns of the indices, and the corresponding sample correlation matrix are

$$10^{-6}\cdot\begin{pmatrix} 523.10 & 517.96 & 240.42 & 261.17 & 576.86 & 496.69 & 491.77 & 939.22 & 217.82 \\ 517.96 & 554.85 & 265.01 & 223.42 & 586.02 & 496.14 & 502.49 & 814.35 & 213.56 \\ 240.42 & 265.01 & 645.86 & 99.41 & 211.91 & 172.23 & 137.85 & 578.67 & 460.57 \\ 261.17 & 223.42 & 99.41 & 277.29 & 248.02 & 226.08 & 217.18 & 467.24 & 204.91 \\ 576.86 & 586.02 & 211.91 & 248.02 & 663.64 & 560.78 & 565.98 & 1039.55 & 162.04 \\ 496.69 & 496.14 & 172.23 & 226.08 & 560.78 & 481.52 & 482.06 & 871.09 & 150.94 \\ 491.77 & 502.49 & 137.85 & 217.18 & 565.98 & 482.06 & 489.54 & 827.20 & 119.56 \\ 939.22 & 814.35 & 578.67 & 467.24 & 1039.55 & 871.09 & 827.20 & 2464.76 & 403.04 \\ 217.82 & 213.56 & 460.57 & 204.91 & 162.04 & 150.94 & 119.56 & 403.04 & 434.42 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.9614 & 0.4136 & 0.6857 & 0.9791 & 0.9897 & 0.9718 & 0.8272 & 0.4569 \\ 0.9614 & 1 & 0.4427 & 0.5696 & 0.9657 & 0.9599 & 0.9642 & 0.6964 & 0.4350 \\ 0.4136 & 0.4427 & 1 & 0.2349 & 0.3237 & 0.3088 & 0.2452 & 0.4586 & 0.8695 \\ 0.6857 & 0.5696 & 0.2349 & 1 & 0.5782 & 0.6187 & 0.5895 & 0.5652 & 0.5904 \\ 0.9791 & 0.9657 & 0.3237 & 0.5782 & 1 & 0.9920 & 0.9930 & 0.8128 & 0.3018 \\ 0.9897 & 0.9599 & 0.3088 & 0.6187 & 0.9920 & 1 & 0.9929 & 0.7996 & 0.3300 \\ 0.9718 & 0.9642 & 0.2452 & 0.5895 & 0.9930 & 0.9929 & 1 & 0.7531 & 0.2593 \\ 0.8272 & 0.6964 & 0.4586 & 0.5652 & 0.8128 & 0.7996 & 0.7531 & 1 & 0.3895 \\ 0.4569 & 0.4350 & 0.8695 & 0.5904 & 0.3018 & 0.3300 & 0.2593 & 0.3895 & 1 \end{pmatrix}$$

It is not a priori clear how well do monthly log returns approximate monthly percentage returns, although they should be reasonably close to each other. However, as can be seen above, it turns out that the sample covariance and correlation matrices for monthly percentage returns and for monthly log returns are very close to each other for this particular set of prices.

(iv) Monthly returns are, generally speaking, likely to be larger than weekly returns, which are likely to be larger than daily returns. We note that the entries of the sample covariance matrices for daily returns are clearly smaller than the entries of the sample covariance matrices for weekly and monthly returns.

The entries of the sample covariance matrices for monthly returns are typically greater than the entries of the sample covariance matrices for weekly returns. However, some of the entries of the sample covariance matrices for weekly returns are larger than the corresponding entries of the sample covariance matrices for monthly returns. These results hold for both percentage and log returns.

The connections between the sample correlation matrices for daily, weekly, and monthly returns are less clear–cut. □

Problem 15: In three months, the value of an asset with spot price \$50 will be either \$60 or \$45. The continuously compounded risk–free rate is 6%. Consider the one period market model with two securities, i.e., cash and the asset, and two states, i.e., asset value equal to \$60 and asset value equal to \$45, in three months.

(i) Find the payoff matrix of this model.

(ii) Is this one period market complete, i.e., is the payoff matrix nonsingular?

(iii) How do you replicate a three months at–the–money put option on this asset, using the cash and the underlying asset?

Solution: The three months one period market model considered here has the following two securities and two states:

Securities:
• cash;
• asset;

Market states:
• asset at \$60 (state ω^1);
• asset at \$45 (state ω^2).

(i) Note that, for a continuously compounded risk–free rate of 6%, the future value in three months of \$1 today is $e^{0.06 \cdot \frac{3}{12}} = 1.015113$.

Then, the payoff matrix of this one period market model corresponding to a \$1 cash position and to an asset position equal to one unit is

$$M_\tau = \begin{pmatrix} 1.015113 & 1.015113 \\ 60 & 45 \end{pmatrix}.$$

(ii) The payoff matrix M_τ is nonsingular since

$$\det(M_\tau) = 1.015113 \cdot 45 - 1.015113 \cdot 60 = -15.23 \neq 0.$$

We conclude that the one period market model is complete.

(iii) The payoff at maturity of a put option is

$$P(T) = \max(K - S(T), 0) = \begin{cases} 0, & \text{if } S(T) \geq K; \\ K - S(T), & \text{if } S(T) < K. \end{cases} \qquad (1.50)$$

From (1.50), we obtain that the values at maturity (i.e., in three months) of at–the–money (ATM) put with strike \$50 on the asset are given by

$$P(1/4) = \max(50 - S(1/4), 0),$$

where $S(1/4)$ denotes the value of the asset in three months, and are as follows:

$$\begin{array}{lll} \text{state } \omega^1: & S(1/4) = 60 \text{ and } & P(1/4) = \max(50-60, 0) = 0; \\ \text{state } \omega^2: & S(1/4) = 45 \text{ and } & P(1/4) = \max(50-45, 0) = 5. \end{array}$$

Thus, the vector value $P_{1/4}$ of the put option in three months is

$$P_{1/4} = (0 \;\; 5). \tag{1.51}$$

Denote by Θ_1 and Θ_2 the cash and asset positions, respectively, in a portfolio replicating the three months ATM put option on the asset, and let $\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$ be the positions vector. Then, the value of the portfolio in three months is

$$V_{1/4} = \Theta^t M_\tau = (\Theta_1 \;\; \Theta_2) \begin{pmatrix} 1.015113 & 1.015113 \\ 60 & 45 \end{pmatrix}. \tag{1.52}$$

For the portfolio to replicate the three months ATM put, $V_{1/4}$ and $P_{1/4}$ must be equal, i.e., $V_{1/4} = P_{1/4}$, which can be written as

$$(\Theta_1 \;\; \Theta_2) \begin{pmatrix} 1.015113 & 1.015113 \\ 60 & 45 \end{pmatrix} = (0 \;\; 5); \tag{1.53}$$

see (1.51) and (1.52). By taking the transpose on both sides of (1.53), we obtain that

$$\begin{pmatrix} 1.015113 & 60 \\ 1.015113 & 45 \end{pmatrix} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}. \tag{1.54}$$

To solve (1.54),[5] recall that the inverse of a 2×2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then, the solution to (1.54) is

$$\begin{aligned} \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} &= \begin{pmatrix} 1.015113 & 60 \\ 1.015113 & 45 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\ &= \frac{1}{-15.2267} \begin{pmatrix} 45 & -60 \\ -1.015113 & 1.015113 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 19.7022 \\ -0.3333 \end{pmatrix}. \end{aligned}$$

We conclude that the portfolio replicating the three months ATM put option on the asset is made of a long cash position of \$19.7022 and a short $0.3333 = \frac{1}{3}$ position in the asset. □

[5] Alternatively, note that (1.54) is equivalent to the linear system

$$\left\{ \begin{array}{lll} 1.015113\, \Theta_1 + 60\, \Theta_2 & = & 0 \\ 1.015113\, \Theta_1 + 45\, \Theta_2 & = & 5 \end{array} \right.$$

and can be solved as such.

Problem 16: In six months, the price of an asset with spot price \$40 will be either \$30, \$35, \$40, \$42, \$45, or \$50. Consider a one period market model with six states in six months corresponding to the six possible values of the asset in six months, and with the following four securities:

- cash;
- asset;
- six months at-the-money call option with strike \$40 on the asset;
- six months at-the-money put option with strike \$40 on the asset.

The continuously compounded risk–free interest rate is constant and equal to 6%.

(i) Find the payoff matrix of this model.

(ii) Is this one period market model complete?

(iii) Are the four securities non–redundant?

Solution: (i) This one period market model has the following four assets and six states in six months:

Securities:
- cash;
- asset;
- six months at-the-money call option with strike \$40 on the asset;
- six months at-the-money put option with strike \$40 on the asset.

States of the market in six months:
- asset price \$30 (state ω^1);
- asset price \$34 (state ω^2);
- asset price \$40 (state ω^3);
- asset price \$42 (state ω^4);
- asset price \$45 (state ω^5);
- asset price \$50 (state ω^6).

The future value in six months of \$1 today corresponding to a continuously compounded 6% constant interest rate is $e^{0.06 \cdot \frac{1}{2}} = e^{0.03}$.

For $j = 1 : 6$, let $S_{j,1/2}$ be the vector of the six possible prices of asset j in six months. The price vectors $S_{1,1/2}$ of cash and $S_{2,1/2}$ of the asset are

$$S_{1,1/2} = \begin{pmatrix} e^{0.03} & e^{0.03} & e^{0.03} & e^{0.03} & e^{0.03} & e^{0.03} \end{pmatrix}; \quad (1.55)$$

$$S_{2,1/2} = \begin{pmatrix} 30 & 35 & 40 & 42 & 45 & 50 \end{pmatrix}. \quad (1.56)$$

Recall that the payoffs at maturity of call and put options are

$$C(T) = \max(S(T) - K, 0) = \begin{cases} S(T) - K, & \text{if } S(T) > K; \\ 0, & \text{if } S(T) \leq K; \end{cases} \quad (1.57)$$

$$P(T) = \max(K - S(T), 0) = \begin{cases} 0, & \text{if } S(T) \geq K; \\ K - S(T), & \text{if } S(T) < K. \end{cases} \quad (1.58)$$

From (1.57), we obtain that the values of the six months ATM call with strike \$40 on the asset are given by

$$C(1/2) = \max(S(1/2) - 40, 0),$$

where $S(1/2)$ denotes the value of the asset in six months, and are as follows:

$$\begin{array}{llll} \text{state } \omega^1: & S(1/2) = 30 & \text{and} & C(1/2) = \max(30-40, 0) = 0; \\ \text{state } \omega^2: & S(1/2) = 35 & \text{and} & C(1/2) = \max(35-40, 0) = 0; \\ \text{state } \omega^3: & S(1/2) = 40 & \text{and} & C(1/2) = \max(40-40, 0) = 0; \\ \text{state } \omega^4: & S(1/2) = 42 & \text{and} & C(1/2) = \max(42-40, 0) = 2; \\ \text{state } \omega^5: & S(1/2) = 45 & \text{and} & C(1/2) = \max(45-40, 0) = 5; \\ \text{state } \omega^6: & S(1/2) = 50 & \text{and} & C(1/2) = \max(50-40, 0) = 10. \end{array}$$

Thus, the vector $S_{3,1/2}$ of the six possible prices of the six months ATM call with strike \$40 on the asset is

$$S_{3,1/2} = (0 \;\; 0 \;\; 0 \;\; 2 \;\; 5 \;\; 10). \tag{1.59}$$

From (1.58), we obtain that the values of the six months ATM put with strike \$40 on the asset are given by

$$P(1/2) = \max(40 - S(1/2), 0),$$

and are as follows:

$$\begin{array}{llll} \text{state } \omega^1: & S(1/2) = 30 & \text{and} & P(1/2) = \max(40-30, 0) = 10; \\ \text{state } \omega^2: & S(1/2) = 35 & \text{and} & P(1/2) = \max(40-35, 0) = 5; \\ \text{state } \omega^3: & S(1/2) = 40 & \text{and} & P(1/2) = \max(40-40, 0) = 0; \\ \text{state } \omega^4: & S(1/2) = 42 & \text{and} & P(1/2) = \max(40-42, 0) = 0; \\ \text{state } \omega^5: & S(1/2) = 45 & \text{and} & P(1/2) = \max(40-45, 0) = 0; \\ \text{state } \omega^6: & S(1/2) = 50 & \text{and} & P(1/2) = \max(40-50, 0) = 0. \end{array}$$

Thus, the vector $S_{4,1/2}$ of the six possible prices of the six months ATM put with strike \$40 on the asset is

$$S_{4,1/2} = (10 \;\; 5 \;\; 0 \;\; 0 \;\; 0 \;\; 0). \tag{1.60}$$

From (1.55), (1.56), (1.59), and (1.60), we conclude that the payoff matrix $M_{1/2}$ of the market model is the following 4×6 matrix:

$$M_{1/2} = \begin{pmatrix} S_{1,1/2} \\ S_{2,1/2} \\ S_{3,1/2} \\ S_{4,1/2} \end{pmatrix} = \begin{pmatrix} e^{0.03} & e^{0.03} & e^{0.03} & e^{0.03} & e^{0.03} & e^{0.03} \\ 30 & 35 & 40 & 42 & 45 & 50 \\ 0 & 0 & 0 & 2 & 5 & 10 \\ 10 & 5 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.61}$$

(ii) This market model is not complete since it has fewer (four) securities than market states (six), and a necessary condition for a one period market model to be complete is that the model has at least as many securities as market states.

(iii) From (1.55), (1.56), (1.59), and (1.60), we obtain that

$$S_{4,1/2} + S_{2,1/2} - S_{3,1/2} = (40 \;\; 40 \;\; 40 \;\; 40 \;\; 40 \;\; 40) = \frac{40}{e^{0.03}} S_{1,1/2}. \tag{1.62}$$

In other words, the price vectors $S_{1,1/2}$, $S_{2,1/2}$, $S_{3,1/2}$, and $S_{4,1/2}$ are not linearly independent. We conclude that, e.g., the ATM put on the asset can be replicated using cash and positions on the asset and on ATM calls on the asset, since

$$S_{4,1/2} = \frac{40}{e^{0.03}} S_{1,1/2} - S_{2,1/2} + S_{3,1/2},$$

and therefore that the ATM put on the asset is a redundant security.[6] □

[6] Note that the redundancy in this model is due to the Put–Call parity; see section 1.2.1 from Stefanica [3] for more details.