Chapter 1

小试牛刀：精选十道题.

1. 两个认沽期权的标的物和到期日均相同, 行权价格分别为\$30和\$20. 如果它们的售价为\$6和\$4, 是否存在套利机会?

2. 2^{29}是一个9位数, 9个数字各不相同. 请问不在其中的那个数字是几?(不必算出2^{29})

3. W_t是维纳过程, 令

$$X_t = \int_0^t W_\tau d\tau.$$

   X_t的分布是怎样的?X_t是鞅吗?

4. 爱丽丝和鲍勃站在一条线段的两端. 鲍勃朝爱丽丝的方向放出50只蚂蚁, 爱丽丝朝鲍勃的方向放出20只蚂蚁, 蚂蚁一只接着一只地朝对方爬去. 两只蚂蚁相遇时, 它们都会掉头继续爬行. 请问最终有多少只蚂蚁爬到鲍勃处, 多少只爬到爱丽丝处? 一共发生了多少次相遇?

5. 找到ρ的取值范围使得

$$\begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix}$$

   是一个相关系数矩阵.

6. 需要生成多少个相互独立的$[0,1]$上的均匀分布随机变量, 才能保证其中至少有一个在0.70和0.72之间的概率为95%?

7. 证明标准正态分布的概率密度函数的积分为1.

8. 假设地球是一个完美的球形, 你站在地球的表面上. 你往南走了1英里, 往东走了1英里, 又往北走了1英里. 你惊喜地发现自己回到了初始位置. 除了北极以外, 你的初始位置还可能是哪里?

9. 求解Ornstein-Uhlenbeck SDE

$$dr_t \;=\; \lambda(\theta - r_t)dt \;+\; \sigma dW_t,$$

   (这个随机微分方程被用在Vasicek模型中.)

10. 编写一个C++函数, 计算第n个斐波那契数.

参考答案

题目 1. 两个认沽期权的标的物和到期日均相同, 行权价格分别为\$30和\$20. 如果它们的售价为\$6和\$4, 是否存在套利机会?

参考答案: 因为行权价格为\$0的认沽期权价值为\$0，我们实际上知道三个不同行权价的期权售价，即

$$P(30) = 6;\ P(20) = 4;\ P(0) = 0,$$

其中$P(K)$为行权价为K的认沽期权的价格.

在平面$(K, P(K))$上，这3个认沽期权分别对应 $(30, 6)$，$(20, 4)$，$(0, 0)$这三个点.我们注意到这三点共线，对应的直线方程为 $P(K) = \frac{2}{3}K$.

我们知道认沽期权的价格应当是关于行权价的严格凹函数，当这一性质被破坏时，就出现了套利机会.

这里的套利机会来源于行权价为20的认沽期权定价过高.根据"低买高卖"的策略，我们应当买入$\frac{2}{3}$份行权价为30的认沽期权，同时卖出1份行权价为20的认沽期权. 为了避免处理小数，我们建立以下投资组合：

- 买入2份行权价格为30的认沽期权；
- 卖出3份行权价格为20的认沽期权.

建立这份投资组合的成本为0，因为由卖出3份行权价格为20的认沽期权和买入2份行权价格为30的认沽期权产生的现金流总和为0.

$$3 \cdot \$4 - 2 \cdot \$6 \ = \ \$0.$$

在期权的到期日T，这份投资组合的价值为

$$V(T) \ = \ 2\max(30 - S(T), 0) - 3\max(20 - S(T), 0).$$

注意到对于任意的标的物价格$S(T)$：

如果$S(T) \geq 30$，则两种期权都没有价值，因此$V(T) = 0$.

如果$20 \le S(T) < 30$，则

$$V(T) \ = \ 2(30 - S(T)) \ > \ 0.$$

如果$0 < S(T) < 20$，则

$$\begin{aligned} V(T) &= 2(30 - S(T)) - 3(20 - S(T)) \\ &= S(T) \\ &> 0. \end{aligned}$$

换言之，我们利用这个套利机会无成本地建立了一份投资组合. 无论在到期时标的物的价格$S(T)$是多少，这份投资组合的回报都非负，并且当$0 < S(T) < 30$ 时, 回报严格为正. □

题目 2. 2^{29}是一个9位数, 9个数字各不相同. 请问不在其中的那个数字是几?(不必算出2^{29})

参考答案: 对于任意正整数n，用$D(n)$表示n各位数的和. 我们知道一个数和它各位数之和的差能被9整除，即

$$9 \mid n - D(n);$$

关于此式的证明，请参考脚注[1].

因此，对于$n = 2^{29}$，我们得到

$$9 \mid 2^{29} - D\left(2^{29}\right). \tag{1.1}$$

[1] 如果n的各位数为a_k，a_{k-1}，$\ldots a_1$，a_0 (从左到右)，则

$$\begin{aligned} n &= a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \ldots a_1 \cdot 10 + a_0; \\ D(n) &= a_k + a_{k-1} + \ldots + a_1 + a_0. \end{aligned}$$

从而，

$$n - D(n) = \sum_{i=0}^{k} a_i \cdot (10^i - 1).$$

由于$10^i - 1$是一个各位数都为9的i位数，所以对于任意$i = 1:k$，$9 \mid 10^i - 1$，因此$9 \mid n - D(n)$.

我们已知2^{29}是九位数，而且九个数字都不同.用x表示缺少的数字，则

$$D\left(2^{29}\right)=\left(\sum_{j=0}^{9} j\right)-x=45-x. \tag{1.2}$$

由(1.1)和(1.2)，可以得到

$$9 \mid 2^{29}-(45-x). \tag{1.3}$$

注意到

$$\begin{aligned} 2^{29} &= 2^5\cdot(2^6)^4 \;=\; 2^5\cdot 64^4 \\ &= 2^5\cdot(63+1)^4 \\ &= 2^5\cdot(63\cdot k+1) \\ &= 2^5\cdot 63\cdot k+2^5, \end{aligned} \tag{1.4}$$

其中k是正整数. [2]

根据(1.4)，我们发现

$$2^{29}-2^5 \;=\; 63\cdot 2^5\cdot k,$$

从而

$$9 \mid 2^{29}-2^5. \tag{1.5}$$

由(1.3)和(1.5)，可以得到

$$\begin{aligned} 9 \quad \mid \quad & (2^{29}-2^5)\;-\;(2^{29}-(45-x)) \\ & =(45-x)-2^5 \\ & =13-x. \end{aligned}$$

[2]容易得到

$$\begin{aligned} (63+1)^4 &= 63^4+4\cdot 63^3+6\cdot 63^2+4\cdot 63+1 \\ &= 63\cdot(63^3+4\cdot 63^2+6\cdot 63+4)+1 \\ &= 63\cdot k+1, \end{aligned}$$

其中 $k=63^3+4\cdot 63^2+6\cdot 63+4$.

由于$9 \mid 13-x$，且x是一位数，我们得到结论$x=4$. 换言之，2^{29}中缺少的数字就是4.

事实上，$2^{29}=536\,870\,912$，正好是九位数，每个数字都不同，且4没有在其中出现. □

题目 3. W_t是维纳过程, 令

$$X_t \;=\; \int_0^t W_\tau d\tau. \tag{1.6}$$

X_t的分布是怎样的?X_t是鞅吗?

参考答案*:* 注意到我们可以把 (1.6) 写作微分形式

$$dX_t \;=\; W_t dt \;=\; W_t dt + 0\, dW_t.$$

所以， X_t是只有漂移项W_t的扩散过程，因此不是鞅.

我们采用分部积分来计算X_t的分布；在5.6节中我们会展示另一种解法.

利用分部积分，我们得到

$$\begin{aligned} X_t &= \int_0^t W_\tau d\tau \\ &= tW_t - \int_0^t \tau dW_\tau \\ &= t\int_0^t dW_\tau - \int_0^t \tau dW_\tau \\ &= \int_0^t (t-\tau) dW_\tau. \end{aligned}$$

我们知道，如果$f(t)$是平方可积的确定性函数，则随机积分$\int_0^t f(\tau)dW_\tau$服从均值为0，方差为$\int_0^t |f(\tau)|^2 d\tau$的正态分布，即，

$$\int_0^t f(\tau)dW_\tau \;\sim\; N\left(0, \int_0^t |f(\tau)|^2 d\tau\right).$$

因此

$$\begin{aligned} X_t &= \int_0^t (t-\tau) dW_\tau \\ &\sim N\left(0, \int_0^t (t-\tau)^2 \, d\tau\right) \\ &= N\left(0, \frac{t^3}{3}\right). \end{aligned}$$

我们得到结论，X_t是均值为0，方差为$\frac{t^3}{3}$的正态随机变量. □

题目 4. 爱丽丝和鲍勃站在一条线段的两端. 鲍勃朝爱丽丝的方向放出50只蚂蚁, 爱丽丝朝鲍勃的方向放出20只蚂蚁, 蚂蚁一只接着一只地朝对方爬去. 两只蚂蚁相遇时, 它们都会掉头继续爬行. 请问最终有多少只蚂蚁爬到鲍勃处, 多少只爬到爱丽丝处? 一共发生了多少次相遇?

参考答案: 想象两只蚂蚁相遇时，它们交换身份.这样，即使两只蚂蚁相遇，它们仍然向相反方向爬去.这样，一定有20只蚂蚁爬到鲍勃处， 50只蚂蚁爬到爱丽丝处.

为了计算蚂蚁相遇的次数，我们假定每只蚂蚁都带着一条讯息.换言之，鲍勃向爱丽丝发出50条信息，由每只蚂蚁携带一条.同样的，爱丽丝向鲍勃发出20条信息，由每只蚂蚁携带一条.进一步假设两只蚂蚁相遇时交换信息.这样，每条信息都总是向前传递.爱丽丝发出的每条消息都要经过50次相遇，鲍勃的每条信息则要经过20次相遇.总共的相遇次数是50乘以20，即1000次. □

题目 5. 找到ρ的取值范围使得

$$\begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix}$$

是一个相关系数矩阵.

参考答案: 对角线为1的对称矩阵是相关系数矩阵, 当且仅当其为半正定矩阵. 因此, 我们要找到ρ的取值范围使以下矩阵半正定.

$$\Omega = \begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix} \tag{1.7}$$

这里我们给出一种用到Sylvester条件的解法. 另外两种解法(一种用到Cholesky分解, 一种基于半正定矩阵的定义)将会在章节5.2中给出.

Sylvester条件指一个矩阵半正定当且仅当其所有主子式大于等于0. 主子式指的行号与列号相同的子方阵的行列式.

式(1.7)中的矩阵Ω有如下主子式:

$$\det(1) = 1; \quad \det(1) = 1; \quad \det(1) = 1;$$

$$\begin{aligned} \det\begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix} &= 0.64; \\ \det\begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix} &= 0.91; \\ \det\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} &= 1-\rho^2; \end{aligned}$$

$$\begin{aligned} \det(\Omega) &= 1 - 0.36\rho - 0.09 - 0.36 - \rho^2 \\ &= 0.55 - 0.36\rho - \rho^2. \end{aligned}$$

因此, 根据Sylvester条件, Ω半正定当且仅当

$$\begin{aligned} 1-\rho^2 &\geq 0; \\ 0.55 - 0.36\rho - \rho^2 &\geq 0, \end{aligned}$$

其等价于 $-1 \leq \rho \leq 1$ 以及

$$\rho^2 + 0.36\rho - 0.55 \leq 0. \tag{1.8}$$

因为式(1.8)的根为-0.9432和0.5832, 我们得到 Ω是相关系数矩阵当且仅当

$$-0.9432 \;\leq\; \rho \;\leq\; 0.5832. \quad \Box \tag{1.9}$$

题目 6. 需要生成多少个相互独立的$[0,1]$上的均匀分布随机变量, 才能保证其中至少有一个在0.70和0.72之间的概率为95%?

参考答案: 记N为使下式成立的最小的随机变量个数

$$P(\text{至少有一个随机变量在}[0.70, 0.72]\text{里}) \;\geq\; 0.95. \tag{1.10}$$

一个在$[0,1]$间均匀分布的随机变量不在区间$[0.70, 0.72]$里的概率为0.98. 因此, N个独立的随机变量均不在$[0.70, 0.72]$里的概率是0.98^N, 即,

$$P(\text{没有随机变量在}[0.70, 0.72]\text{里}) \;=\; 0.98^N.$$

注意到

$$\begin{aligned} & P(\text{至少有一个随机变量在}[0.70, 0.72]\text{里}) \\ = \;& 1 - P(\text{没有随机变量在}[0.70, 0.72]\text{里}) \\ = \;& 1 - (0.98)^N. \end{aligned} \tag{1.11}$$

由式(1.10)和式(1.11), 可得N是使下式成立的最小整数

$$1 - (0.98)^N \;\geq\; 0.95,$$

其等价于

$$\begin{aligned} & (0.98)^N \leq 0.05 \\ \Longleftrightarrow \quad & N \ln(0.98) \leq \ln(0.05) \\ \Longleftrightarrow \quad & N \geq \frac{\ln(0.05)}{\ln(0.98)} \approx 148.28 \\ \Longleftrightarrow \quad & N = 149. \end{aligned}$$

综上, 为了保证至少有一个变量在0.70和0.72之间的概率为95%, 至少要生成149个$[0,1]$间均匀分布的随机变量. □

题目 7. 证明标准正态分布的概率密度函数的积分为1.

参考答案: 标准正态分布的概率密度函数为 $\frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}$. 我们想要证明

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt \;=\; 1,$$

做变量替换$t = \sqrt{2}x$, 上式可以写作

$$I \;=\; \int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}. \tag{1.12}$$

接下来我们用极坐标系来证明式(1.12). 因为x是积分变量, 我们可以把I写成关于另一个积分变量y的积分 $I = \int_{-\infty}^{\infty} e^{-y^2} dy$.

所以,[3]

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dxdy \qquad (1.13) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy. \end{aligned}$$

对最后一个积分使用极坐标系变换 $x = r\cos\theta$和$y = r\sin\theta$,

[3]注意到等式(1.13)的严格推导要用到Fubini定理; 这个技术步骤几乎不会被面试官问到.

其中$r \in [0,\ \infty)$, $\theta \in [0,\ 2\pi)$. 因为$dxdy = rd\theta dr$, 我们得到

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dxdy \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-\left(r^2\cos^2\theta + r^2\sin^2\theta\right)}\ d\theta dr \\ &= \int_0^{\infty}\int_0^{2\pi} r\ e^{-r^2}\ d\theta dr \qquad (1.14) \\ &= \int_0^{\infty} 2\pi\ r\ e^{-r^2}\ dr \\ &= 2\pi\ \lim_{t\to\infty}\int_0^t r\ e^{-r^2}\ dr \\ &= 2\pi\ \lim_{t\to\infty}\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^t \\ &= \pi; \end{aligned}$$

注意到式(1.14)用到了等式 $\cos^2\theta + \sin^2\theta = 1$对任意实数$\theta$成立.

因为$I > 0$, 所以$I = \sqrt{\pi}$, 我们证明了式(1.12). □

题目 8. 假设地球是一个完美的球形, 你站在地球的表面上. 你往南走了1英里, 往东走了1英里, 又往北走了1英里. 你惊喜地发现自己回到了初始位置. 除了北极以外, 你的初始位置还可能是哪里?

参考答案: 除北极之外, 满足这个条件的位置还有无数个.

在地球表面上, 南极附近有一圈周长为一英里的纬线. 从这条纬线上的任意一点P出发, 往东(或西)走一英里, 你将回到点P. 如果你从点P以北一英里处出发, 往南一英里走到点P. 接着往东走一英里回到点P(绕纬线走一圈), 最后往北一英里回到初始位置.

考虑到点P为这条特殊纬线上的任意一点, 初始位置为点P以北一英里, 我们有无数个满足条件的初始位置.

事实上, 这样的特殊纬线也有无数条. 南极附近任意周长为$1/k$英里(k为正整数)的纬线都可以. 从这样的纬线以北一英里出发, 往南一英里走到特殊纬线上的一个点, 往东一

英里回到这个点(绕纬线走k圈), 最终往北一英里回到初始位置. □

题目 9. 求解Ornstein-Uhlenbeck SDE

$$dr_t \ = \ \lambda(\theta - r_t)dt \ + \ \sigma dW_t, \tag{1.15}$$

(这个随机微分方程被用在Vasicek模型中.)

参考答案: 我们把式(1.15)写作

$$dr_t + \lambda r_t dt \ = \ \lambda\theta dt + \sigma dW_t. \tag{1.16}$$

在式(1.16)的等号两边同时乘以$e^{\lambda t}$, 我们得到

$$e^{\lambda t} dr_t + \lambda e^{\lambda t} r_t dt \ = \ \lambda\theta e^{\lambda t} dt + \sigma e^{\lambda t} dW_t,$$

其等价于

$$d\left(e^{\lambda t} r_t\right) \ = \ \lambda\theta e^{\lambda t} dt + \sigma e^{\lambda t} dW_t. \tag{1.17}$$

对式(1.17)从0到t积分,有

$$\begin{aligned} e^{\lambda t} r_t - r_0 &= \lambda\theta \int_0^t e^{\lambda s} ds + \sigma \int_0^t e^{\lambda s} dW_s \\ &= \theta\left(e^{\lambda t} - 1\right) + \sigma \int_0^t e^{\lambda s} dW_s \,. \end{aligned}$$

所以Ornstein-Uhlenbeck SDE的解为

$$\begin{aligned} r_t &= e^{-\lambda t} r_0 + e^{-\lambda t}\theta\left(e^{\lambda t} - 1\right) + \sigma e^{-\lambda t} \int_0^t e^{\lambda s} dW_s \\ &= e^{-\lambda t} r_0 + \theta\left(1 - e^{-\lambda t}\right) + \sigma \int_0^t e^{-\lambda(t-s)} dW_s. \end{aligned}$$

注意到无论起始位置r_0在哪, 随机过程r_t都会朝θ均值回归. 这是因为非随机函数$f(s)$的随机积分$\int_0^t f(s) dW_s$的期望值为0, 故而

$$E\left[\int_0^t e^{-\lambda(t-s)} dW_s\right] = 0,$$

所以

$$E[r_t] \;=\; e^{-\lambda t} r_0 + \theta\left(1 - e^{-\lambda t}\right).$$

所以,

$$\lim_{t\to\infty} E[r_t] \;=\; \theta. \quad \square$$

题目 10. 编写一个C++函数, 计算第n个斐波那契数.

参考答案: 斐波那契数列$(F_n)_{n\geq 0}$由以下递归式定义:

$$F_{n+2} \;=\; F_{n+1} + F_n, \;\; \forall\, n \geq 0,$$

同时$F_0 = 0$, $F_1 = 1$.

```
//recursive implementation
int fib(int n) {
    if (n == 0 || n == 1) return n;
    else {
        return fib(n-1) + fib(n-2);
    }
}

//iterative implementation
int fib(int n ){
    if (n == 0 || n == 1) return n;
    int prev = 0, last = 1, temp;
    for (int i = 2; i <= n; ++i) {
        temp = last;
        last = prev + last;
        prev = temp;
    }
    return last;
}

//tail recursive implementation
int fib(int n, int last = 1, int prev = 0)
{
```

```
    if (n == 0) return prev;
    if (n == 1) return last;
        return fib(n-1, last+prev, last);
}
```

Chapter 2

First Look: Ten Questions.

1. Put options with strikes 30 and 20 on the same underlying asset and with the same maturity are trading for \$6 and \$4, respectively. Can you find an arbitrage?

2. The number 2^{29} has 9 digits, all different. Without computing 2^{29}, find the missing digit.

3. Let W_t be a Wiener process, and let

$$X_t = \int_0^t W_\tau d\tau.$$

   What is the distribution of X_t? Is X_t a martingale?

4. Alice and Bob stand at opposite ends of a straight line segment. Bob sends 50 ants towards Alice, one after another. Alice sends 20 ants towards Bob. All ants travel along the straight line segment. Whenever two ants collide, they simply bounce back and start traveling in the opposite direction. How many

ants reach Bob and how many ants reach Alice? How many ant collisions take place?

5. Find all the values of ρ such that

$$\begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix}$$

is a correlation matrix.

6. How many independent random variables uniformly distributed on $[0, 1]$ should you generate to ensure that there is at least one between 0.70 and 0.72 with probability 95%?

7. Show that the probability density function of the standard normal integrates to 1.

8. Assume the Earth is perfectly spherical and you are standing somewhere on its surface. You travel exactly 1 mile south, then 1 mile east, then 1 mile north. Surprisingly, you find yourself back at the starting point. If you are not at the North Pole, where can you possibly be?!

9. Solve the Ornstein-Uhlenbeck SDE

$$dr_t \;=\; \lambda(\theta - r_t)dt \;+\; \sigma dW_t,$$

which is used, e.g., in the Vasicek model for interest rates.

10. Write a C++ function that computes the n-th Fibonacci number.

Solutions

Question 1. Put options with strikes 30 and 20 on the same underlying asset and with the same maturity are trading for \$6 and \$4, respectively. Can you find an arbitrage?

Answer: Since the value of a put option with strike 0 is \$0, we in fact know the prices of put options with three different strikes, i.e.,

$$P(30) = 6;\; P(20) = 4;\; P(0) = 0,$$

where $P(K)$ denotes the value of a put option with strike K.

In the plane $(K, P(K))$, these option values correspond to the points $(30, 6)$, $(20, 4)$, and $(0, 0)$, which are on the line $P(K) = \frac{2}{3}K$.

This contradicts the fact that put options are strictly convex functions of strike price, and creates an arbitrage opportunity.

The arbitrage comes from the fact that the put with strike 20 is overpriced. Using a "buy low, sell high" strategy, we could buy (i.e., go long) $\frac{2}{3}$ put options with strike 30, and sell (i.e., go short) 1 put option with strike 20. To avoid fractions, we set up the following portfolio:

- long 2 puts with strike 30;
- short 3 puts with strike 20.

This portfolio is set up at no initial cost, since the cash flow generated by selling 3 puts with strike 20 and buying 2 puts with strike 30 is \$0:

$$3 \cdot \$4 - 2 \cdot \$6 \;=\; \$0.$$

At the maturity T of the options, the value of the portfolio is

$$V(T) \;=\; 2\max(30 - S(T), 0) - 3\max(20 - S(T), 0).$$

Note that $V(T)$ is nonnegative for any value $S(T)$ of the underlying asset:

If $S(T) \geq 30$, then both put options expire worthless, and $V(T) = 0$.

If $20 \leq S(T) < 30$, then

$$V(T) \;=\; 2(30 - S(T)) \;>\; 0.$$

If $0 < S(T) < 20$, then

$$\begin{aligned} V(T) &= 2(30 - S(T)) - 3(20 - S(T)) \\ &= S(T) \\ &> 0. \end{aligned}$$

In other words, we took advantage of the existing arbitrage opportunity by setting up, at no initial cost, a portfolio with nonnegative payoff at T regardless of the price $S(T)$ of the underlying asset, and with a strictly positive payoff if $0 < S(T) < 30$. □

Question 2. The number 2^{29} has 9 digits, all different. Without computing 2^{29}, find the missing digit.

Answer: For any positive integer n, denote by $D(n)$ the sum of the digits of n. Recall that the difference between a number and the sum of its digits is divisible by 9, i.e.,

$$9 \mid n - D(n);$$

see the footnote below[1] for details.

---

[1] If the digits of n are $a_k, a_{k-1}, \ldots a_1, a_0$ (from left to right), then

$$\begin{aligned} n &= a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \ldots a_1 \cdot 10 + a_0; \\ D(n) &= a_k + a_{k-1} + \ldots + a_1 + a_0. \end{aligned}$$

Hence,

$$n - D(n) = \sum_{i=0}^{k} a_i \cdot (10^i - 1).$$

Thus, for $n = 2^{29}$, it follows that

$$9 \mid 2^{29} - D\left(2^{29}\right). \tag{2.1}$$

We are given that 2^{29} has 9 digits, and that all 9 digits are different. Denote by x the missing digit. Then,

$$D\left(2^{29}\right) = \left(\sum_{j=0}^{9} j\right) - x = 45 - x. \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$9 \mid 2^{29} - (45 - x). \tag{2.3}$$

Note that

$$\begin{aligned} 2^{29} &= 2^5 \cdot (2^6)^4 \;=\; 2^5 \cdot 64^4 \\ &= 2^5 \cdot (63+1)^4 \\ &= 2^5 \cdot (63 \cdot k + 1) \\ &= 2^5 \cdot 63 \cdot k + 2^5, \end{aligned} \tag{2.4}$$

where k is a positive integer.[2]

From (2.4), we find that

$$2^{29} - 2^5 \;=\; 63 \cdot 2^5 \cdot k,$$

and therefore

$$9 \mid 2^{29} - 2^5. \tag{2.5}$$

Since $10^i - 1$ is an i-digit number with all digits equal to 9, it follows that $9 \mid 10^i - 1$, for all $i = 1 : k$, and therefore $9 \mid n - D(n)$.

[2]It is easy to see that

$$\begin{aligned} (63+1)^4 &= 63^4 + 4 \cdot 63^3 + 6 \cdot 63^2 + 4 \cdot 63 + 1 \\ &= 63 \cdot (63^3 + 4 \cdot 63^2 + 6 \cdot 63 + 4) + 1 \\ &= 63 \cdot k + 1, \end{aligned}$$

where $k = 63^3 + 4 \cdot 63^2 + 6 \cdot 63 + 4$.

From (2.3) and (2.5), it follows that

$$\begin{aligned} 9 \quad | \quad & (2^{29} - 2^5) \; - \; (2^{29} - (45 - x)) \\ & = (45 - x) - 2^5 \\ & = 13 - x. \end{aligned}$$

Since $9 \mid 13 - x$ and x is a digit, we conclude that $x = 4$. In other words, we identified that x, the missing digit from 2^{29}, must be 4.

Indeed, $2^{29} = 536\,870\,912$, i.e., 2^{29} has 9 digits, all different, and 4 is not a digit of 2^{29}. □

Question 3. Let W_t be a Wiener process, and let

$$X_t \;=\; \int_0^t W_\tau d\tau. \tag{2.6}$$

What is the distribution of X_t? Is X_t a martingale?

Answer: Note that we can rewrite (2.6) in differential form as

$$dX_t \;=\; W_t dt \;=\; W_t dt + 0\, dW_t.$$

Then, X_t is a diffusion process with only drift part W_t, and therefore X_t is not a martingale.

We use integration by parts to find the distribution of X_t; a different solution can be found in Section 6.6.

By applying integration by parts, we obtain that

$$\begin{aligned} X_t \quad &= \quad \int_0^t W_\tau d\tau \\ &= \quad tW_t - \int_0^t \tau dW_\tau \\ &= \quad t\int_0^t dW_\tau - \int_0^t \tau dW_\tau \\ &= \quad \int_0^t (t - \tau) dW_\tau. \end{aligned}$$

Recall that, if $f(t)$ is a deterministic square integrable function, then the stochastic integral $\int_0^t f(\tau)dW_\tau$ is a normal random variable of mean 0 and variance $\int_0^t |f(\tau)|^2 d\tau$, i.e.,

$$\int_0^t f(\tau)dW_\tau \sim N\left(0, \int_0^t |f(\tau)|^2 d\tau\right).$$

Thus,

$$\begin{aligned} X_t &= \int_0^t (t-\tau)dW_\tau \\ &\sim N\left(0, \int_0^t (t-\tau)^2 \, d\tau\right) \\ &= N\left(0, \frac{t^3}{3}\right). \end{aligned}$$

We conclude that X_t is a normal random variable of mean 0 and variance $\frac{t^3}{3}$. □

Question 4. Alice and Bob stand at opposite ends of a straight line segment. Bob sends 50 ants towards Alice, one after another. Alice sends 20 ants towards Bob. All ants travel along the straight line segment. Whenever two ants collide, they simply bounce back and start traveling in the opposite direction. How many ants reach Bob and how many ants reach Alice? How many ant collisions take place?

Answer: Imagine that when two ants meet, they switch identities. Hence, even after a collision, two ants are traveling in two opposite directions. It follows that 20 ants reach Bob, while 50 ants reach Alice.

To calculate the number of ant collisions, imagine that each ant carries a message. In other words, Bob sends 50 messages to Alice, one message per ant. Similarly, Alice sends 20 messages to Bob, one message per ant. Furthermore, imagine that the two ants swap messages when they collide. Then a message always makes forward progress. Each of Alice's messages goes through 50 ant collisions. Each of Bob's messages goes through 20 ant collisions. The total number of collisions is 50 times 20, which is 1000 collisions. $\square$

Question 5. Find all the values of ρ such that

$$\begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix}$$

is a correlation matrix.

Answer: A symmetric matrix with diagonal entries equal to 1 is a correlation matrix if and only if the matrix is symmetric positive semidefinite. Thus, we need to find all the values of ρ such that the matrix

$$\Omega = \begin{pmatrix} 1 & 0.6 & -0.3 \\ 0.6 & 1 & \rho \\ -0.3 & \rho & 1 \end{pmatrix} \qquad (2.7)$$

is symmetric positive semidefinite.

We give a short solution using Sylvester's criterion. Two more solutions, one using the Cholesky decomposition, and another one based on the definition of symmetric positive semidefinite matrices will be given in Section 6.2.

Recall from Sylvester's criterion that a matrix is symmetric positive semidefinite if and only if all its principal minors are greater than or equal to 0. Also, recall that the principal minors of a matrix are the determinants of all the square matrices obtained by eliminating the same rows and columns from the matrix. In particular, the matrix Ω from (2.7) has the following principal minors:

$$\det(1) = 1; \quad \det(1) = 1; \quad \det(1) = 1;$$

$$\det\begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix} = 0.64;$$
$$\det\begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix} = 0.91;$$
$$\det\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = 1-\rho^2;$$

$$\begin{aligned} \det(\Omega) &= 1 - 0.36\rho - 0.09 - 0.36 - \rho^2 \\ &= 0.55 - 0.36\rho - \rho^2. \end{aligned}$$

Thus, it follows from Sylvester's criterion that Ω is a symmetric positive semidefinite matrix if and only if

$$\begin{aligned} 1-\rho^2 &\geq 0; \\ 0.55 - 0.36\rho - \rho^2 &\geq 0, \end{aligned}$$

which is equivalent to $-1 \leq \rho \leq 1$ and

$$\rho^2 + 0.36\rho - 0.55 \leq 0. \qquad (2.8)$$

Since the roots of the quadratic equation corresponding to (2.8) are -0.9432 and 0.5832, we conclude that the matrix Ω is symmetric positive semidefinite, and therefore a correlation matrix, if and only if

$$-0.9432 \leq \rho \leq 0.5832. \quad \Box \qquad (2.9)$$

Question 6. How many independent random variables uniformly distributed on $[0,1]$ should you generate to ensure that there is at least one between 0.70 and 0.72 with probability 95%?

Answer: Denote by N the smallest number of random variables you should generate such that

$$P(\text{at least one r.v. in } [0.70, 0.72]) \geq 0.95. \qquad (2.10)$$

The probability that a random variable uniformly distributed on $[0,1]$ is not in the interval $[0.70, 0.72]$ is 0.98. Thus, the probability that none of the N independent variables are in $[0.70, 0.72]$ is 0.98^N, i.e.,

$$P(\text{no r.v. in } [0.70, 0.72]) \;=\; 0.98^N.$$

Note that

$$\begin{aligned} & P(\text{at least one r.v. in } [0.70, 0.72]) \\ = \;& 1 - P(\text{no r.v. in } [0.70, 0.72]) \\ = \;& 1 - (0.98)^N. \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we find that N is the smallest integer such that

$$1 - (0.98)^N \;\geq\; 0.95,$$

which is equivalent to

$$\begin{aligned} & (0.98)^N \leq 0.05 \\ \Longleftrightarrow \quad & N \ln(0.98) \leq \ln(0.05) \\ \Longleftrightarrow \quad & N \geq \frac{\ln(0.05)}{\ln(0.98)} \approx 148.28 \\ \Longleftrightarrow \quad & N = 149. \end{aligned}$$

We conclude that at least 149 uniform random variables on $[0,1]$ must be generated in order to have 95% confidence that at least one of the random variables is between 0.70 and 0.72. □

Question 7. Show that the probability density function of the standard normal integrates to 1.

Answer: The probability density function of the standard normal variable is $\frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}$. We want to show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt \;=\; 1,$$

which, using the substitution $t = \sqrt{2}x$, can be written as

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{2.12}$$

We prove (2.12) by using polar coordinates. Since x is just an integrating variable, we can also write the integral I in terms of another integrating variable, denoted by y, as $I = \int_{-\infty}^{\infty} e^{-y^2} dy$.

Then,[3]

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy. \end{aligned} \tag{2.13}$$

We use the polar coordinates transformation $x = r\cos\theta$ and $y = r\sin\theta$, with $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$, to evaluate the last integral. Since $dxdy = rd\theta dr$, we obtain

[3]Note that Fubini's theorem is needed for a rigorous derivation of the equality (2.13); this technical step is rarely required by the interviewer.

that

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy \\
&= \int_0^{\infty}\int_0^{2\pi} r\, e^{-\left(r^2\cos^2\theta + r^2\sin^2\theta\right)}\, d\theta dr \\
&= \int_0^{\infty}\int_0^{2\pi} r\, e^{-r^2}\, d\theta dr \qquad (2.14) \\
&= \int_0^{\infty} 2\pi\, r\, e^{-r^2}\, dr \\
&= 2\pi \lim_{t\to\infty} \int_0^{t} r\, e^{-r^2}\, dr \\
&= 2\pi \lim_{t\to\infty} \left(-\frac{1}{2} e^{-r^2} \right)\Big|_0^t \\
&= \pi;
\end{aligned}
$$

note that (2.14) follows from the equality $\cos^2\theta + \sin^2\theta = 1$ for any real number θ.

Since $I > 0$, $I = \sqrt{\pi}$, which is what we wanted to prove; see (2.12). □

Question 8. Assume the Earth is perfectly spherical and you are standing somewhere on its surface. You travel exactly 1 mile south, then 1 mile east, then 1 mile north. Surprisingly, you find yourself back at the starting point. If you are not at the North Pole, where can you possibly be?!

Answer: There are infinitely many locations, aside from the North Pole, that have this property.

Somewhere near the South Pole, there is a latitude that has a circumference of one mile. In other words, if you are at this latitude and start walking east (or west), in one mile you will be back exactly where you started from. If you instead start at some point one mile north of this latitude, your journey will take you one mile south to this special latitude, then one mile east "around the globe" and finally one mile north right back to wherever you started from. Moreover, there are infinitely many points on the Earth that are one mile north of this special latitude, where you could start your journey and eventually end up exactly where you started.

We are still not finished! There are infinitely many special latitudes as well; namely, you could start at any point one mile north of the latitude that has a circumference of $1/k$ miles, where k is a positive integer. Your journey will take you one mile south to this special latitude, then one mile east looping "around the globe" k times, and finally one mile north right back to where you started from. □

Question 9. Solve the Ornstein-Uhlenbeck SDE

$$dr_t \;=\; \lambda(\theta - r_t)dt \;+\; \sigma dW_t, \tag{2.15}$$

which is used, e.g., in the Vasicek model for interest rates.

Answer: We can rewrite (2.15) as

$$dr_t + \lambda r_t dt \;=\; \lambda\theta dt + \sigma dW_t. \tag{2.16}$$

By multiplying (2.16) on both sides by the integrating factor $e^{\lambda t}$, we obtain that

$$e^{\lambda t} dr_t + \lambda e^{\lambda t} r_t dt \;=\; \lambda\theta e^{\lambda t} dt + \sigma e^{\lambda t} dW_t,$$

which is equivalent to

$$d\left(e^{\lambda t} r_t\right) \;=\; \lambda\theta e^{\lambda t} dt + \sigma e^{\lambda t} dW_t. \tag{2.17}$$

By integrating (2.17) from 0 to t, it follows that

$$\begin{aligned} e^{\lambda t} r_t - r_0 \;&=\; \lambda\theta \int_0^t e^{\lambda s} ds + \sigma \int_0^t e^{\lambda s} dW_s \\ &=\; \theta\left(e^{\lambda t} - 1\right) + \sigma \int_0^t e^{\lambda s} dW_s\,. \end{aligned}$$

By solving for r_t, we find that the solution to the Ornstein-Uhlenbeck SDE is

$$\begin{aligned} r_t &= e^{-\lambda t} r_0 + e^{-\lambda t}\theta\left(e^{\lambda t} - 1\right) + \sigma e^{-\lambda t}\int_0^t e^{\lambda s} dW_s \\ &= e^{-\lambda t} r_0 + \theta\left(1 - e^{-\lambda t}\right) + \sigma\int_0^t e^{-\lambda(t-s)} dW_s. \end{aligned}$$

Note that the process r_t is mean reverting to θ, regardless of the starting point r_0. To see this, recall that the expected value of the stochastic integral $\int_0^t f(s)dW_s$ of a non-random function $f(s)$ is 0. Then,

$$E\left[\int_0^t e^{-\lambda(t-s)} dW_s\right] = 0,$$

and therefore

$$E[r_t] \;=\; e^{-\lambda t} r_0 + \theta\left(1 - e^{-\lambda t}\right).$$

Thus,

$$\lim_{t\to\infty} E[r_t] \;=\; \theta. \quad \square$$

Question 10. Write a C++ function that computes the n-th Fibonacci number.

Answer: The Fibonacci numbers $(F_n)_{n\geq 0}$ are given by the following recurrence:

$$F_{n+2} \;=\; F_{n+1} + F_n, \;\; \forall\, n \geq 0,$$

with $F_0 = 0$ and $F_1 = 1$.

```
//recursive implementation
int fib(int n) {
    if (n == 0 || n == 1) return n;
    else {
        return fib(n-1) + fib(n-2);
```

```
    }
}

//iterative implementation
int fib(int n ){
    if (n == 0 || n == 1) return n;
    int prev = 0, last = 1, temp;
    for (int i = 2; i <= n; ++i) {
        temp = last;
        last = prev + last;
        prev = temp;
    }
    return last;
}

//tail recursive implementation
int fib(int n, int last = 1, int prev = 0)
{
    if (n == 0) return prev;
    if (n == 1) return last;
        return fib(n-1, last+prev, last);
}
```