§25. Random Walks in Two Dimensions

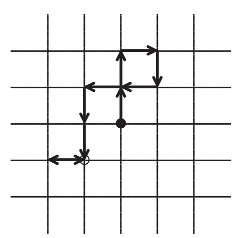
Let $\mathbb{Z}^2$ denote all points in the Euclidean plane with integer coordinates. Specifically, $\mathbb{Z}^2 = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z}\}$. We seek to define a random walk on $\mathbb{Z}^2$, which we will denote by $(W_n^{(2)} : n \geq 0)$. Here the superscript indicates the dimension of the process and we will use $W_n^{(1)}$ to denote a (one dimensional) simple symmetric random walk on $\mathbb{Z}$, formerly simply called W_n.

Each point $(x, y) \in \mathbb{Z}^2$ has four nearest neighbors, north, south, east, and west, that are a distance 1 from (x, y), namely: $(x, y + 1)$, $(x, y - 1)$, $(x + 1, y)$, and $(x - 1, y)$. We define a Markov chain $(W_n^{(2)} : n \geq 0)$ on state space $\mathbb{Z}^2$ as follows. Take $W_0^{(2)} = (0, 0)$, so the process starts at the origin. Then, given $W_n^{(2)} = (x, y)$, $W_{n+1}^{(2)}$ is one of $W_n^{(2)}$'s four nearest neighbors — each with probability $\frac{1}{4}$. Stated differently,

$$W_{n+1}^{(2)} = \begin{cases} W_n^{(2)} + (+1, 0) & \text{with probability } \frac{1}{4} \\ W_n^{(2)} + (-1, 0) & \text{with probability } \frac{1}{4} \\ W_n^{(2)} + (0, +1) & \text{with probability } \frac{1}{4} \\ W_n^{(2)} + (0, -1) & \text{with probability } \frac{1}{4}, \end{cases}$$

where the increment is independent of the process up to time n. We will let X_n and Y_n denote the first and second coordinates of $W_n^{(2)}$, respectively, so $W_n^{(2)} = (X_n, Y_n)$. Figure 25.1 shows a realization of the first 10 steps of such a walk. Here $W_{10} = (-1, -1)$, so $X_{10} = Y_{10} = -1$ (the '◦' in the figure).

Figure 25.1. An illustrative realization of $W_n^{(2)}$ for $0 \leq n \leq 10$.



This defines a *simple symmetric random walk on* $\mathbb{Z}^2$. The state space $\mathbb{Z}^2$ is clearly irreducible under this process and, like the simple symmetric random walk on $\mathbb{Z}$, it has period 2. To verify this latter statement, note that $X_n + Y_n$ changes parity with each step. Like the simple symmetric random walk on $\mathbb{Z}$, we can only have $W_n^{(2)} = (0, 0)$ if n is even. Our goal here is to show that the simple symmetric random walk on $\mathbb{Z}^2$ is recurrent. We have

Theorem 25.2. $P[W_{2n}^{(2)} = (0, 0)] = (P[W_{2n}^{(1)} = 0])^2$*, so* $W_n^{(2)}$ *is recurrent.*

Proof. Let $\mathbf{u} = (\frac{1}{2}, \frac{1}{2})$ and $\mathbf{v} = (\frac{1}{2}, -\frac{1}{2})$. Then $\mathbf{u} + \mathbf{v} = (1, 0)$, $-\mathbf{u} - \mathbf{v} = (-1, 0)$, $\mathbf{u} - \mathbf{v} = (0, 1)$, and $-\mathbf{u} + \mathbf{v} = (0, -1)$. So if A and B are independent with

$P[A = \pm 1] = P[B = \pm 1] = \frac{1}{2}$, then $A\mathbf{u} + B\mathbf{v}$ is either $(1,0)$, $(-1,0)$, $(0,1)$, or $(0,-1)$ — each with probability $\frac{1}{4}$. Let $(A_i : i \geq 1)$ and $(B_i : i \geq 1)$ be independent iid sequences with $P[A_i = \pm 1] = P[B_i = \pm 1] = \frac{1}{2}$. By the above discussion,

$$W_n^{(2)} \;=\; (0,0) + (A_1\mathbf{u} + B_1\mathbf{v}) + \cdots + (A_n\mathbf{u} + B_n\mathbf{v})$$

is a SSRW on $\mathbb{Z}^2$ starting at the origin. But

$$\begin{aligned} W_{2n}^{(2)} = (0,0) &\iff (A_1 + \cdots + A_{2n})\mathbf{u} + (B_1 + \cdots + B_{2n})\mathbf{v} = (0,0) \\ &\iff A_1 + \cdots + A_{2n} = 0 \text{ and } B_1 + \cdots + B_{2n} = 0. \end{aligned}$$

Where the second ‘ $\iff$ ’ holds because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors. Now $W_n = A_1 + \cdots + A_n$ and $W_n^* = B_1 + \cdots + B_n$ define independent SSRWs on $\mathbb{Z}$ starting at 0 so

$$P[W_{2n}^{(2)} = (0,0)] \;=\; P[W_{2n} = 0 \text{ and } W_{2n}^* = 0] \;=\; (P[W_{2n}^{(1)} = 0])^2.$$

As for the assertion of recurrence, from (24.13), we see that

$$P[W_{2n}^{(2)} = (0,0)] \;=\; \frac{b_n^2}{n}$$

where $b_n^2 \to b^2 > 0$ as $n \to \infty$. Since $\sum_n \frac{1}{n}$ diverges, we get that $W_n^{(2)}$ is recurrent. □

This beautiful argument is surprisingly simple. Presently we offer a second proof that is more computational in nature. It too is nice because at the heart is an appealing combinatorial identity.

Alternative Proof. Fix some number n. Recall that X_i and Y_i denote the first and second coordinates of $W_i^{(2)}$, respectively, so $W_i^{(2)} = (X_i, Y_i)$. With each step in the process, either X_i or Y_i (but not both) change by ± 1. Let K (which is random) denote the number of steps i, with $0 \leq i < 2n$ where it is the first coordinate that changes, i.e. where $X_{i+1} = X_i \pm 1$ and $Y_{i+1} = Y_i$. Then $0 \leq K \leq 2n$, specifically $K \sim \text{Binomial}\left(2n, \frac{1}{2}\right)$. If K is odd, then X_{2n} is odd and cannot be 0 — therefore $W_{2n}^{(2)}$ cannot be (0,0). That is, $P[W_{2n}^{(2)} = (0,0) \cap K = k] = 0$ for odd k. Hence

$$\begin{aligned} P[W_{2n}^{(2)} = (0,0)] &= \sum_{k=0}^{2n} P[W_{2n}^{(2)} = (0,0) \cap K = k] \\ &= \sum_{k=0}^{n} P[W_{2n}^{(2)} = (0,0) \cap K = 2k]. \end{aligned}$$

If $2k$ of the $2n$ steps involve the X coordinate then k of those steps must be to the right ($X_{i+1} = X_i + 1$) and k to the left to have $X_{2n} = 0$. Similarly, $n - k$ of the $2n - 2k$ changes of the Y coordinate must be up ($Y_{i+1} = Y_i + 1$), and $n - k$ down, to have $Y_{2n} = 0$. To specify a walk of length $2n$ that begins and ends at $(0, 0)$ for which the X coordinate changes $2k$ times we must therefore: (i) specify which $2k$ of the $2n$ steps involve a change of the X coordinate; (ii) specify which k of those $2k$ steps involve a step to the right; and (iii) specify which $n-k$ of the remaining $2n-2k$ steps involve a step up. It follows that there are $\binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}$ such walks, each with probability $\left(\frac{1}{4}\right)^{2n} = \left(\frac{1}{2}\right)^{4n}$, yielding

$$P[W_{2n}^{(2)} = (0,0) \cap K = 2k] \;=\; \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{2}\right)^{4n},$$

and thus

$$P[W_{2n}^{(2)} = (0,0)] \;=\; \left(\frac{1}{2}\right)^{4n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{2k}.$$

Now $\binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{2k} = \binom{2n}{n}\binom{n}{k}^2$ (work out both sides, they're both $\frac{(2n)!}{[k!(n-k)!]^2}$), so

$$\begin{aligned} P[W_{2n}^{(2)} = (0,0)] &= \left(\frac{1}{2}\right)^{4n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2 \\ &= \left(\frac{1}{2}\right)^{4n}\binom{2n}{n}^2 \\ &= \left[\left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\right]^2 \;=\; P[W_{2n}^{(1)} = 0]^2. \end{aligned}$$

The second equality rests on the combinatoric identity $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$. To see this, suppose we wish to select n bottles of wine from among $2n$ bottles. There are $\binom{2n}{n}$ ways to do this. Now suppose n of the $2n$ bottles are red wine and n are white wine, and let's organize the count a different way. Let k denote how many bottles of red we select, so we also select $n-k$ bottles of white. There are $\binom{n}{k}$ ways to select the k reds and $\binom{n}{n-k}$ ways to select the remaining $n-k$ whites. Hence, once k is selected, there are $\binom{n}{k}\cdot\binom{n}{n-k} = \binom{n}{k}^2$ ways to select the n wines in such a way that exactly k are red. Clearly we must have $0 \le k \le n$, so the number of ways to select the n wines is also given by $\sum_{k=0}^{n}\binom{n}{k}^2$. We have partitioned the count according to how many bottles of red are selected. $\square$

EXERCISES

1. In the alternative proof of Theorem 25.2 show that, given $K = 2k$, X_{2n} and Y_{2n} are conditionally independent. That is, show that

$$\begin{aligned} &P[X_{2n} = x, Y_{2n} = y \mid K = 2k] \\ &\qquad = P[X_{2n} = x \mid K = 2k] \cdot P[Y_{2n} = y \mid K = 2k]. \end{aligned}$$

2. For the two-dimensional random walk $W_n^{(2)} = (X_n, Y_n)$, show that the expected squared distance from the origin after n steps is $E[X_n^2 + Y_n^2] = n$.

3. Use that $W_n^{(2)} = (0,0) \implies X_n = 0$ and that the SSRW on $\mathbb{Z}$ is not positive recurrent to argue that $W_n^{(2)}$ is not positive recurrent.

4. Define a random walk, call it $\widetilde{W}$, on $\mathbb{Z}^2$ as follows. $\widetilde{W}_0 = (0,0)$ and

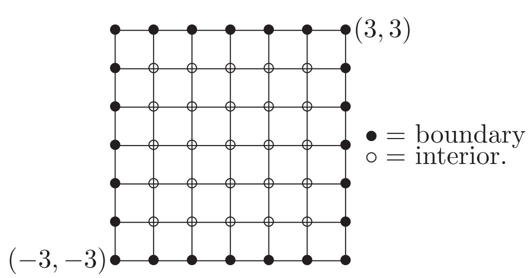
$$\widetilde{W}_{n+1} = \begin{cases} \widetilde{W}_n + (1,1) & \text{with probability } \frac{1}{4} \\ \widetilde{W}_n + (1,-1) & \text{with probability } \frac{1}{4} \\ \widetilde{W}_n + (-1,1) & \text{with probability } \frac{1}{4} \\ \widetilde{W}_n + (-1,-1) & \text{with probability } \frac{1}{4}, \end{cases}$$

where the increment is independent of the process up to time n. Show that $P[\widetilde{W}_n = 0] = P[W_n^{(2)} = 0]$.

• *Discrete Harmonic Functions.* Fix positive integer k and let

$$C = \{(m,n) : m \in \mathbb{Z}, n \in \mathbb{Z}, |m| \le k, |n| \le k\}.$$

Then C is a $(2k+1) \times (2k+1)$ square portion of $\mathbb{Z}^2$ with the origin at its center. Let ∂C denote the *boundary* of C (this is standard notation): $\partial C = \{(m,n) \in C : |m| = k \text{ or } |n| = k\}$. The *interior* of C, denoted C°, is everything else in C, so $C^\circ = C \setminus \partial C$. We show this here for $k = 3$:



A *discrete harmonic* function on C is a function $f : C \to \mathbb{R}$ such that

$$\text{for } (m,n) \in C^\circ,$$
$$f(m,n) = \frac{f(m,n+1) + f(m,n-1) + f(m+1,n) + f(m-1,n)}{4}. \qquad (*)$$

In words, the value of $f(m,n)$ at an interior point is the average of the values at that point's four nearest $\mathbb{Z}^2$ neighbors.

5. Suppose $g : \partial C \to \mathbb{R}$ is any function, so g assigns numbers to the boundary points of C. Show that there is a discrete harmonic function $f : C \to \mathbb{R}$ with $f(m,n) = g(m,n)$ for all points $(m,n) \in \partial C$. Hint: Let $(W_i^{(2)}(m,n) : i \geq 0)$ be a SSRW on $\mathbb{Z}^2$ starting at (m,n) and let G denote the value of $g()$ at the location where this walk first hits ∂C.

6. Show that the harmonic function $f(m,n)$ satisfying this boundary condition is unique. Hint: follow a procedure similar to problem 1, §24.

7. For any function $f(x,y)$, let $\Delta_x f(x,y) = f(x+1,y) - f(x,y)$, and put $\Delta_x^2 f(x,y) = \Delta_x f(x,y) - \Delta_x f(x-1,y)$. Similarly define $\Delta_y^2 f(x,y)$, keeping x fixed and varying y. Show that the discrete harmonic condition $(*)$ is equivalent to $\Delta_x^2 f(m,n) + \Delta_y^2 f(m,n) = 0$. (A function $f(x,y)$ defined on an open region $R \subset \mathbb{R}^2$ is harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ at each point in the region.)

§26. Random Walks in Three Dimensions

Here we study the recurrence/transience of the simple symmetric random walk in three dimensions. The state space for this walk is $\mathbb{Z}^3 = \{(x,y,z) : x \in \mathbb{Z},\ y \in \mathbb{Z},\ z \in \mathbb{Z}\}$. We denote this walk by $W_n^{(3)} = (X_n, Y_n, Z_n)$, $n \geq 0$, where $W_0^{(3)} = (0,0,0)$ and, for $n \geq 0$,

$$W_{n+1}^{(3)} = \begin{cases} W_n^{(3)} + (+1,0,0) & \text{with probability } \frac{1}{6} \\ W_n^{(3)} + (-1,0,0) & \text{with probability } \frac{1}{6} \\ W_n^{(3)} + (0,+1,0) & \text{with probability } \frac{1}{6} \\ W_n^{(3)} + (0,-1,0) & \text{with probability } \frac{1}{6} \\ W_n^{(3)} + (0,0,+1) & \text{with probability } \frac{1}{6} \\ W_n^{(3)} + (0,0,-1) & \text{with probability } \frac{1}{6}, \end{cases}$$

where the increment is independent of the process up to time n. We seek to establish that:

Theorem 26.1. *For some number D, $P[W_{2n}^{(3)} = (0,0,0)] \leq \frac{D}{n^{3/2}}$ so $W_n^{(3)}$ is transient.*

Remarks. Like the one and two dimensional cases, if n is odd, $W_n^{(3)}$ cannot be $(0,0,0)$ because the parity of $X_i + Y_i + Z_i$ changes with each step. The transience conclusion follows from the fact that $\sum_n \frac{1}{n^{3/2}} < \infty$ together with Theorem 20.5 or, if you wish, the Borel-Cantelli Lemma. One might expect, based on the two dimensional case, that $P[W_{2n}^{(3)} = (0,0,0)] = (P[W_{2n}^{(1)} = 0])^3$. This is not