2. Same for $\mathcal{T} = \{0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00\}$.

§38. Itô's Formula for Brownian Motion

Here we develop an elementary version of Itô's formula in the context of Brownian motion. Suppose $f(t, b)$ is a continuous function of two variables. Then, as we have discussed, $X_t = f(t, B_t)$ describes a new stochastic process with continuous paths. For the discussion that follows to be rigorous, we impose at this juncture some technical restrictions. For convenience, we restrict attention to the X_t process on a finite time interval $[0, T]$. Additionally, we require that:

$$\begin{array}{c}\text{(i) the partial derivatives } f_{tt},\ f_{tb},\ f_{bbb} \text{ all exist and are continuous;} \\ \text{(ii) for some } h > 0, \quad E\left[(f_{bb}(t, B_t))^2\right] \le h \text{ for all } t \in [0, T].\end{array} \tag{38.1}$$

Condition (ii) is rather technical, but not terribly onerous. Presently we prove:

THEOREM 38.2 (ITÔ'S FORMULA). *If the function $f(t, b)$ satisfies (38.1), then on the interval $[0, T]$ the process $X_t = f(t, B_t)$ has dynamics given by*

$$dX_t = \left(f_t(t, B_t) + \frac{1}{2} f_{bb}(t, B_t)\right)dt + f_b(t, B_t)dB_t.$$

NOTATIONAL REMARK. There is some room for notational confusion here, as X_t denotes the value of the stochastic process X at time t while f_t denotes the partial derivative of f with respect to t. We will use capital letters to denote stochastic processes and lower case letters to denote functions and real variables. This should mitigate any confusion.

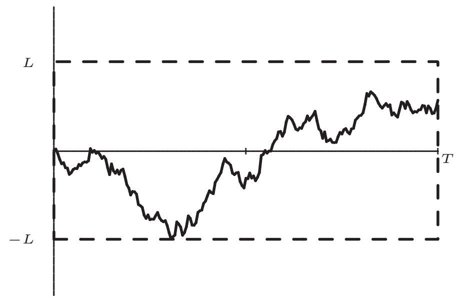
EXAMPLE 38.3. In Example 37.8 we have $f(t, b) = b^2 - t$, so $f_t(t, b) = -1$, $f_b(t, b) = 2b$, and $f_{bb}(t, b) = 2$. This gives

$$dX_t = \left(-1 + \frac{1}{2} \cdot 2\right)dt + 2B_t dB_t = 2B_t dB_t.$$

Here the drift term (M_t in (37.3)) is zero — this is a *zero drift* process.

PROOF OF THEOREM 38.2. For any Brownian path B let $L = \max\{|B_t| : 0 \le t \le T\}$. Then the rectangle $\mathcal{R}(T, L) = \{(t, b) : 0 \le t \le T \text{ and } |b| \le L\}$ contains the path. Since f's derivatives in (38.1) are continuous they are bounded in absolute value in this rectangle (see Figure 38.4). Letting C denote the least such upper bound, we observe that C is a random variable as it depends on ω through the path B that is associated with ω.

Figure 38.4. f's derivatives are bounded by C in $\mathcal{R}(T, L)$.



Fix $0 \le u < v \le T$ and let $\mathcal{T} = \{t_i\}$ be a partition of $[u, v]$. Using standard estimation techniques for functions of two variables (see §45), we get that

$$\begin{aligned}\Delta X_{t_i} &= f(t_{i+1}, B_{t_{i+1}}) - f(t_i, B_{t_i}) \\ &= f_t(t_i, B_{t_i})\Delta t_i + f_b(t_i, B_{t_i})\Delta B_{t_i} + \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta B_{t_i}^2 \\ &\quad + \mathit{Error}_i,\end{aligned}$$

where $|\mathit{Error}_i| \le C \cdot (\Delta t_i^2 + \Delta t_i |\Delta B_{t_i}| + |\Delta B_{t_i}|^3)$. Here the term $\epsilon(t, \Delta t)$ as defined in (37.3) is given by

$$\begin{aligned}\epsilon(t_i, \Delta t_i) &= \Delta X_{t_i} - \left[\left(f_t(t_i, B_{t_i}) + \frac{1}{2} f_{bb}(t_i, B_{t_i})\right)\Delta t_i + f_b(t_i, B_{t_i})\Delta B_{t_i}\right] \\ &= \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta B_{t_i}^2 \; - \; \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta t_i \; + \; \mathit{Error}_i. \qquad (38.5)\end{aligned}$$

We will show that, as $||\mathcal{T}|| \to 0$,

$$\sum_i \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta B_{t_i}^2 \;\xrightarrow{\text{p}}\; \int_u^v \frac{1}{2} f_{bb}(t, B_t)\, dt, \qquad (38.6)$$

and

$$\sum_i |\mathit{Error}_i| \;\le\; C \cdot \sum_i \left(\Delta t_i^2 + \Delta t_i |\Delta B_{t_i}| + |\Delta B_{t_i}|^3\right) \;\xrightarrow{\text{p}}\; 0. \qquad (38.7)$$

Note that the right side of (38.6) is a perfectly legitimate Riemann integral, as, for each ω, the integrand is a continuous function of t. If we combine (38.6) with the fact that

$$\sum_i \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta t_i \to \int_u^v \frac{1}{2} f_{bb}(t, B_t)\, dt \quad \text{as} \quad ||\mathcal{T}|| \to 0,$$

we see that

$$\left|\sum_i \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta B_{t_i}^2 \;-\sum_i \frac{1}{2} f_{bb}(t_i, B_{t_i})\Delta t_i\right| \xrightarrow{\text{p}} 0 \quad \text{as} \quad ||\mathcal{T}|| \to 0.$$

($X_m \xrightarrow{\text{p}} X,\ Y_m \xrightarrow{\text{p}} X \implies |X_m - Y_m| \xrightarrow{\text{p}} 0$, see exercises.) This together with (38.7) shows that $\sum_i \epsilon(t_i, \Delta t_i) \xrightarrow{\text{p}} 0$, as required (refer to (38.5)).

First we turn our attention to the error terms in (38.7). Note that

$$\sum_i \Delta {t_i}^2 \;\leq\; ||\mathcal{T}|| \sum_i \Delta t_i \;=\; ||\mathcal{T}|| \cdot (v-u) \;\to\; 0, \quad \text{as } ||\mathcal{T}|| \to 0.$$

That was easy. As for the second error term,

$$\begin{aligned}\sum_i \Delta t_i |\Delta B_{t_i}| \;&\leq\; \max\{|\Delta B_{t_i}| : 0 \leq i < n\} \cdot \sum_i \Delta t_i \\ &=\; \max\{|\Delta B_{t_i}|\} \cdot (v-u),\end{aligned}$$

so it will suffice to show that

$$\max\{|\Delta B_{t_i}|\} \;\to\; 0, \quad \text{as } ||\mathcal{T}|| \to 0. \tag{38.8}$$

But B_t is a continuous function of t. Hence, on any closed and bounded interval $[u, v]$ it is *uniformly continuous* (see Appendix II). That is, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|B_t - B_{t'}| < \epsilon$ whenever $t, t' \in [u, v]$ with $|t - t'| < \delta$. Note that $\delta = \delta(\omega)$ is random because it will depend on the Brownian path $B = B(\omega)$. Now fix any $\epsilon > 0$ and let $\delta > 0$ correspond to that ϵ. Then $||\mathcal{T}|| < \delta \implies \Delta t_i < \delta \implies |B_{t_i} - B_{t_{i+1}}| < \epsilon$. Because this holds for each individual i, we see that $||\mathcal{T}|| < \delta \implies \max\{|\Delta B_{t_i}|\} < \epsilon$. This establishes (38.8).

For the third error term, use Lemma 38.9 (stated below) as applied to $H_t = 1$ to get that $\sum_i \Delta B_{t_i}^2 \xrightarrow{\text{p}} v - u$ as $||\mathcal{T}|| \to 0$. Then, as $||\mathcal{T}|| \to 0$,

$$\sum_i |\Delta B_{t_i}^3| \;\leq\; \max\{|\Delta B_{t_i}|\} \cdot \sum_i \Delta B_{t_i}^2 \;\xrightarrow{\text{p}}\; 0,$$

as $X_m \xrightarrow{\text{p}} 0,\ Y_m \xrightarrow{\text{p}} c \implies X_m Y_m \xrightarrow{\text{p}} 0$ (see exercises). Collecting these three error terms yields that

$$\sum_i \left(\Delta t_i^2 + \Delta t_i |\Delta B_{t_i}| + |\Delta B_{t_i}|^3\right) \;\xrightarrow{\text{p}}\; 0,$$

as $||\mathcal{T}|| \to 0$. This is enough to get (38.7), for $Y_n \xrightarrow{\text{p}} 0 \implies XY_n \xrightarrow{\text{p}} 0$ (see, e.g., Theorem 8.6).

Now we turn our attention to establishing (38.6), which is the crux of the matter. We have:

LEMMA 38.9. *If the process H_t satisfies* (36.1) *and has bounded second moment for $t \in [0,T]$ then*

$$\sum_{i=0}^{n-1} H_{t_i} \Delta B_{t_i}^2 \xrightarrow{\text{p}} \int_u^v H_t\, dt \quad \textit{as} \quad ||\mathcal{T}|| \to 0.$$

Statement (38.6) is simply this lemma as applied to $H_t = \frac{1}{2} f_{bb}(t, B_t)$, which has bounded second moment for $t \in [0,T]$ by virtue of (38.1).

PROOF OF LEMMA 38.9. For partition $\mathcal{T} = \{t_0, \ldots, t_n\}$ of $[u,v]$, let

$$\begin{aligned} Z_{\mathcal{T}} &= \sum_i H_{t_i} \Delta B_{t_i}^2 - \sum_i H_{t_i} \Delta t_i \\ &= \sum_i H_{t_i} [\Delta B_{t_i}^2 - \Delta t_i] \\ &= \sum_i H_i M_i, \end{aligned}$$

where $H_i = H_{t_i}$ and $M_i = \Delta B_{t_i}^2 - \Delta t_i$. We will show that $Z_{\mathcal{T}} \xrightarrow{\text{p}} 0$ as $||\mathcal{T}|| \to 0$. Since

$$\sum_{i=0}^{n-1} H_{t_i} \Delta t_i \ \to \ \int_u^v H_t\, dt \quad \text{as } ||\mathcal{T}|| \to 0,$$

the lemma will follow ($X_m - Y_m \xrightarrow{\text{p}} 0,\ Y_m \xrightarrow{\text{p}} Y \implies X_m \xrightarrow{\text{p}} Y$, see exercises). Note that this holds because the left side is a Riemann sum approximation of the integral on the right side. (Again, the integrand is a continuous function of t.) We will show that $EZ_{\mathcal{T}} = 0$ and that $\operatorname{Var} Z_{\mathcal{T}} \to 0$ as $||\mathcal{T}|| \to 0$. Then, by Chebyshev's inequality, $P[|Z_{\mathcal{T}}| \geq \epsilon] \leq \frac{\operatorname{Var} Z_{\mathcal{T}}}{\epsilon^2} \to 0$, establishing that $Z_{\mathcal{T}} \xrightarrow{\text{p}} 0$.

Now $E[\Delta B_{t_i}] = 0$, so $E[\Delta B_{t_i}^2] = \operatorname{Var} \Delta B_{t_i} = \Delta t_i$ and $EM_i = 0$. Furthermore, H_i depends on B_t only through time t_i while M_i depends on the Brownian increment on $[t_i, t_{i+1}]$, so H_i and M_i are independent and $E[H_i M_i] = EH_i \cdot EM_i = 0$. By linearity of expectation, $EZ_{\mathcal{T}} = 0$. Regarding the variance, if $0 \leq i < j$, then

$$\begin{aligned} E[H_i M_i H_j M_j] &= E[H_i M_i H_j] \cdot E[M_j] \\ &= E[H_i M_i H_j] \cdot 0 \ = \ 0 \ = \ E[H_i M_i] \cdot E[H_j M_j]. \end{aligned}$$

The first equality holds because $H_i M_i H_j$ depends on the Brownian path up to time t_j, while M_j depends on the Brownian increment over the interval $[t_j, t_{j+1}]$ — $H_i M_i H_j$ and M_j are therefore independent. We have that $H_i M_i$ and $H_j M_j$ are uncorrelated if $i \neq j$, so

$$\begin{aligned} \operatorname{Var} Z_{\mathcal{T}} \ = \ \sum_i \operatorname{Var}[H_i M_i] \ = \ \sum_i E[(H_i M_i)^2] \ &= \ \sum_i E[H_i^2] E[M_i^2] \\ &\leq \ h \sum_i \operatorname{Var}[M_i], \end{aligned} \tag{38.10}$$

where h is the hypothesized bound for EH_t^2. The second equality above holds because H_iM_i is mean 0; the third because H_i^2 and M_i^2 are independent. The inequality uses that M_i is mean 0. Now $\text{Var}\,[M_i] = \text{Var}\,[\Delta B_{t_i}^2 - \Delta t_i] = \text{Var}\,[\Delta B_{t_i}^2]$, and

$$\Delta B_{t_i}^2 \;\sim\; (\text{Normal}\,(0, \Delta t_i))^2 \;\sim\; \Delta t_i (\text{Normal}\,(0,1))^2,$$

so

$$\text{Var}\, M_i \;=\; \Delta t_i^2 \text{Var}\,[(\text{Normal}\,(0,1))^2] \;=\; 2\Delta t_i^2,$$

using that $(\text{Normal}\,(0,1))^2$ has the chi-squared distribution with 1 degree of freedom which has variance 2. Then

$$\sum_i \text{Var}\,[M_i] \;=\; \sum_i 2\Delta t_i^2 \;\leq\; 2||\mathcal{T}|| \cdot \sum_i \Delta t_i \;=\; 2||\mathcal{T}|| \cdot (v-u) \;\to\; 0,$$

as $||\mathcal{T}|| \to 0$. Combining this with (38.10) gives that $\text{Var}\, Z_{\mathcal{T}} \to 0$ as $||\mathcal{T}|| \to 0$. This concludes the proof of Lemma 38.9 and Theorem 38.2. □

Remark. Lemma 38.9 is sometimes informally paraphrased as "$dB^2 = dt$."

EXERCISES

• In problems 1 – 3, verify the assertions made in the proof of Theorem 38.2 and Lemma 38.9.

1. $X_m \xrightarrow{\text{p}} X,\ Y_m \xrightarrow{\text{p}} X \implies |X_m - Y_m| \xrightarrow{\text{p}} 0.$

2. $X_m \xrightarrow{\text{p}} 0,\ Y_m \xrightarrow{\text{p}} c \implies X_mY_m \xrightarrow{\text{p}} 0.$

3. $X_m - Y_m \xrightarrow{\text{p}} 0,\ Y_m \xrightarrow{\text{p}} Y \implies X_m \xrightarrow{\text{p}} Y.$

• In problems 4 – 7, compute the dynamics of $X_t = f(t,b)$.

4. $f(t,b) = b$.

5. $f(t,b) = tb$.

6. $f(t,b) = tb^2$.

7. $f(t,b) = e^b$ (geometric Brownian motion).

8. Determine a function $g(t)$ so that $f(t,b) = g(t)e^b$ generates a zero drift process.

§39. Application: The Black-Scholes PDE and Formula

Here we study a classic application of Itô's formula to develop the Black-Scholes formula which values a call option on an underlying stock. It is customary in finance to model the evolution of a stock price as a random process following *geometric Brownian motion*:

$$S_t \;=\; s_0 e^{\mu t + \sigma B_t}, \tag{39.1}$$