1.1 Discrete Probability

1. There are M green apples and N red apples in a basket. We take apples out randomly one by one until all the apples left in the basket are red. What is the probability that at the moment we stop the basket is empty?

2. A fair coin is tossed n times. What is the expected product of the number of heads and the number of tails?

3. If $x_1, x_2, \ldots, x_9$ is a random arrangement of numbers 1, 2, ..., 9 around a circle, what is the probability that $\sum_{i=1}^{9} |x_{i+1} - x_i|$ is minimized? (Here, $x_{10} = x_1$.)

4. There are 1000 green balls and 3000 red balls in container A, and 3000 green balls and 1000 red balls in container B. You take half of the balls from A at random and transfer them to B. Then you take one ball from B at random. What is the probability that this ball is green?

5. A robot performs coin tossing. It is poorly designed, it produces a lot of sounds, lights, and vapors, and it takes one hour to toss a coin. Yet in the end, when the coin finally lands, it somehow has equal probability of showing heads and tails.

   Two scientists, A and B, enjoy observing this robot and, by analyzing its unusual and faulty behavior, they became fairly decent at guessing whether the coin will land heads or tails half an hour before the coin is released from the robot's hand. The scientist A has 80% chance of successfully predicting the outcome, while the scientist B is successful 60% of the time.

The robot started its routine, and the scientist A predicts the coin will land tails. The scientist B predicts the coin will land heads. Can you calculate the probability that the coin will land heads?

6. A player chooses a number $k \leq 52$ and the top k cards are drawn one by one from a properly shuffled standard deck of 52 cards. The player wins if the last drawn card is an Ace and if there is exactly one more Ace among the cards drawn. Which k should the player choose to maximize the chance of winning in this game?

7. Let N be a random variable whose values are positive integers. Prove that

$$\mathbb{E}[N] \;=\; \sum_{i=0}^{\infty} \mathbb{P}(N > i)\,.$$

8. Each box of cereal contains a coupon. If there are p kinds of coupons, how many boxes of cereal have to be bought on average to obtain at least one coupon of each kind?

9. You roll a fair n-sided die repeatedly and sum the outcomes. What is the expected number of rolls until the sum is a multiple of n for the first time?

10. Is it possible to have two non-fair 6-sided dice, with sides numbered 1 through 6, with a uniform sum probability?

11. Consider 2^n players of equal skill[1] playing a game where the players are paired off against each other at random. The 2^{n-1} winners are again paired off randomly, and so on until a single winner remains.

[1] The probability of winning a game between any two players is $\frac{1}{2}$ for each player.

Chapter 2

Solutions

2.1 Discrete Probability

Question 1. There are M green apples and N red apples in a basket. We take apples out randomly one by one until all the apples left in the basket are red. What is the probability that at the moment we stop the basket is empty?

Answer: Let A be the event that at the moment we stop the basket is empty. Since we must take apples out until all the green apples are out, the only way we would have to empty the entire basket in order to take all the green apples out is if the last apple in the basket is green.

Hence, we can now modify our random experiment in the following way: Denote the green apples by g_1, g_2, $\ldots$, g_M and the red apples by r_1, r_2, $\ldots$, r_N. Instead of taking the apples out of the basket randomly one by one until all the apples left in the basket are red, we will take the apples out until no apple is left in the basket. The sample space Ω for our experiment now becomes the set of all the permutations of the set

$$S = \{g_1, g_2, \ldots, g_M, r_1, r_2, \ldots, r_N\}.$$

The size of Ω is $(M+N)!$. The probability function $\mathbb{P}$ assigns the value $\frac{1}{(M+N)!}$ to each of the outcomes.

Note that A is the set of all permutations from Ω whose last entry is from $G = \{g_1, g_2, \ldots, g_M\}$. The number of permutations whose last entry is from G is $M \cdot (M+N-1)!$, since there are M ways to choose the last entry and $(M+N-1)!$ ways to permute the remaining entries.

We conclude that the probability of the event A is

$$\mathbb{P}(A) = \frac{(M+N-1)! \cdot M}{(M+N)!} = \frac{M}{M+N}. \qquad (2.1)$$

In retrospect, it would have been easier to derive (2.1) by noticing that, in the modified random experiment, the last ball in the basket is equally likely to be any of the balls in S, hence, it is green with probability $\frac{M}{M+N}$. □

Question 2. A fair coin is tossed n times. What is the expected product of the number of heads and the number of tails?

Answer: Let $X_1, X_2, \ldots, X_n$ be random variables that correspond to the individual coin tosses in the following way: the value of X_i is 1, if the i-th toss is head, and zero otherwise. Then, the number of heads, H_n, and the number of tails, T_n, are random variables that can be expressed in terms of $X_1, \ldots, X_n$ as

$$\begin{aligned} H_n &= X_1 + X_2 + \cdots + X_n, \text{ and} \\ T_n &= n - (X_1 + X_2 + \cdots + X_n). \end{aligned}$$

Using linearity, the expectation of the product $H_n T_n$ of the number of heads and the number of tails is given by

$$\begin{aligned} \mathbb{E}[H_n T_n] &= n\,\mathbb{E}[X_1 + \cdots + X_n] \\ &\quad - \mathbb{E}\left[(X_1 + \cdots + X_n)^2\right]. \qquad (2.2) \end{aligned}$$

Since the coin is fair, $\mathbb{E}[X_i] = \frac{1}{2}$ and $\mathbb{E}\left[X_i^2\right] = \frac{1}{2}$ for all $i = 1 : n$, and, since X_j and X_k are independent for $j \neq k$,

$$\mathbb{E}[X_j X_k] \;=\; \mathbb{E}[X_j]\,\mathbb{E}[X_k] \;=\; \frac{1}{4}, \quad \forall\ 1 \leq j \neq k \leq n.$$

Then,

$$\mathbb{E}[X_1 + \cdots + X_n] \;=\; \sum_{i=1}^{n} \mathbb{E}[X_i] \;=\; \frac{n}{2}; \tag{2.3}$$

$$\begin{aligned}
&\mathbb{E}\left[(X_1 + \cdots + X_n)^2\right] \\
=\;& \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right] \;+\; \sum_{1 \leq j \neq k \leq n} \mathbb{E}[X_j X_k] \\
=\;& n \cdot \frac{1}{2} \;+\; (n^2 - n) \cdot \frac{1}{4} \\
=\;& \frac{n^2 + n}{4}.
\end{aligned} \tag{2.4}$$

From (2.2–2.4), we conclude that

$$\mathbb{E}[H_n T_n] \;=\; \frac{n^2}{2} - \frac{n^2 + n}{4} \;=\; \frac{n^2 - n}{4}. \quad \square$$

Remark. The argument above is very similar to the derivation of formulas for the expected value and variance of binomial random variables. If we use these formulas (instead of deriving them), then the solution can be made a bit shorter. We first need to observe that H_n has a binomial distribution with parameters n and $p = \frac{1}{2}$. Its expected value and variance satisfy the equations $\mathbb{E}[H_n] = np$ and $\mathrm{var}(H_n) = np(1-p)$. Therefore

$$\begin{aligned}
\mathbb{E}[H_n T_n] \;&=\; \mathbb{E}[H_n(n - H_n] = n\mathbb{E}[H_n] - \mathbb{E}\left[H_n^2\right] \\
&=\; n^2 p - np(1-p) - n^2 p^2 \\
&=\; \left(n^2 - n\right) p(1-p).
\end{aligned}$$

Question 3. If $x_1, x_2, \ldots, x_9$ is a random arrangement of numbers 1, 2, ..., 9 around a circle, what is the probability that $\sum_{i=1}^{9} |x_{i+1} - x_i|$ is minimized? (Here, $x_{10} = x_1$.)

Answer: We will use the following notations:

$$\begin{aligned} \overrightarrow{x} &= (x_1, \ldots, x_9)\,; \\ S(\overrightarrow{x}) &= \sum_{i=1}^{9} |x_{i+1} - x_i|\,. \end{aligned}$$

Since the given sum does not change if the numbers are rotated around the circle, we can assume without any loss of generality that $x_1 = 1$. Let $1 < k \leq 9$ such that $x_k = 9$. Using the triangle inequality,[1] we obtain that

$$\begin{aligned} S(\overrightarrow{x}) &= \sum_{i=1}^{k-1} |x_{i+1} - x_i| + \sum_{i=k}^{9} |x_{i+1} - x_i| \\ &\geq \left|\sum_{i=1}^{k-1} (x_{i+1} - x_i)\right| + \left|\sum_{i=k}^{9} (x_{i+1} - x_i)\right| \\ &= |x_k - x_1| + |x_{10} - x_k| \\ &= |9 - 1| + |1 - 9| \\ &= 18. \end{aligned}$$

The minimal value of $S(\overrightarrow{x}) = \sum_{i=1}^{9} |x_{i+1} - x_i|$ is 18, which is achieved if and only if

$$\begin{aligned} \sum_{i=1}^{k-1} |x_{i+1} - x_i| &= \left|\sum_{i=1}^{k-1} (x_{i+1} - x_i)\right|; \\ \sum_{i=k}^{9} |x_{i+1} - x_i| &= \left|\sum_{i=k}^{9} (x_{i+1} - x_i)\right|, \end{aligned}$$

which in turn happens if and only if the sequence ($x_1 = 1$, x_2, ..., $x_k = 9$) is increasing and the sequence ($x_k = 9$, x_{k+1}, ..., $x_{10} = 1$) is decreasing.

[1] The triangle inequality states that $|a| + |b| \geq |a + b|$ for any (real or complex) numbers a and b.

Since $x_1 = 1$, there is a total of 8! arrangements of the numbers 2, 3, ..., 9 around the circle, all of them being equally likely. Let G be the set of all arrangements of $x_1 = 1, x_2, \ldots, x_9$ around the circle that minimize the described sum. The probability that the sum $\sum_{i=1}^{9} |x_{i+1} - x_i|$ is minimized is equal to $\frac{|G|}{8!}$. For each $k \in \{2, 3, \ldots, 9\}$, let $G_k \subseteq G$ be the set of such arrangements with $x_k = 9$. Once the set of numbers $S_{2,k-1} = \{x_2, \ldots, x_{k-1}\}$ is fixed, the set $S_{k+1,9} = \{x_{k+1}, \ldots, x_9\}$ is also fixed. Moreover, the numbers $x_2, \ldots, x_{k-1}$ must form the unique permutation of $S_{2,k-1}$ in which the numbers are in increasing order. Similarly, the numbers $x_{k+1}, \ldots, x_9$ must form the unique permutation of $S_{k+1,9}$ in which the numbers are in decreasing order. Therefore, an arrangement in G_k is uniquely determined by the set $S_{2,k-1}$.

We conclude that $|G_k|$ is equal to the number of ways in which the elements of $S_{2,k-1}$ can be chosen. The number of such choices is $\binom{7}{k-2}$. Therefore,

$$\begin{aligned} |G| &= \sum_{k=2}^{9} |G_k| = \sum_{k=2}^{9} \binom{7}{k-2} \\ &= \binom{7}{0} + \binom{7}{1} + \cdots + \binom{7}{7} = 2^7. \end{aligned}$$

Thus, the probability that the sum $\sum_{i=1}^{9} |x_{i+1} - x_i|$ is minimized is equal to

$$\frac{|G|}{8!} = \frac{2^7}{8!}. \quad \square$$

Question 4. There are 1000 green balls and 3000 red balls in container A, and 3000 green balls and 1000 red balls in container B. You take half of the balls from A at random and transfer them to B. Then you take one ball from B at random. What is the probability that this ball is green?

Answer: We need to calculate the probability of the event G that the selected ball is green. Denote by A the event that the selected ball was originally from container A and by A^C the event that the selected ball was originally from container B. Note that $\mathbb{P}(A) = \frac{1}{3}$ and $\mathbb{P}\left(A^C\right) = \frac{2}{3}$, since 2000 balls were added from container A to the 4000 balls in container B.

Conditioned on the event that the selected ball is from A, the probability of the ball being green is $\frac{1}{4}$, that is, $\mathbb{P}(G|A) = \frac{1}{4}$. Conditioned on the event that the selected ball is from A^C, the probability of the ball being green is $\frac{3}{4}$, that is, $\mathbb{P}\left(G|A^C\right) = \frac{3}{4}$.

We conclude that

$$\begin{aligned} \mathbb{P}(G) &= \mathbb{P}(G|A)\,\mathbb{P}(A) \; + \; \mathbb{P}\left(G|A^C\right)\,\mathbb{P}\left(A^C\right) \\ &= \frac{1}{4}\cdot\frac{1}{3} \; + \; \frac{3}{4}\cdot\frac{2}{3} \\ &= \frac{7}{12}. \quad \square \end{aligned}$$

Question 5. A robot performs coin tossing. It is poorly designed, it produces a lot of sounds, lights, and vapors, and it takes one hour to toss a coin. Yet in the end, when the coin finally lands, it somehow has equal probability of showing heads and tails.

Two scientists, A and B, enjoy observing this robot and, by analyzing its unusual and faulty behavior, they became fairly decent at guessing whether the coin will land heads or tails half an hour before the coin is released from the robot's hand. The scientist A has 80% chance of successfully predicting the outcome, while the scientist B is successful 60% of the time.

The robot started its routine, and the scientist A predicts the coin will land tails. The scientist B predicts the coin will land heads. Can you calculate the probability that the coin will land heads?

Answer: The correct answer to this problem is "No, I cannot." We will build two probability spaces that satisfy the conditions of the problem, but have different probabilities of heads when conditioned on the described event.

Denote by E the event that A predicts T and B predicts H, and by L_H the event that the coin lands heads. We will build two probability spaces $(\Omega_1, \mathbb{P}_1)$ and $(\Omega_2, \mathbb{P}_2)$, such that

$$\mathbb{P}_1\left(L_H \middle| E\right) \neq \mathbb{P}_2\left(L_H \middle| E\right).$$

First, we consider the following sample space

$$\Omega_1 = \{HA_1B_1, TA_1B_1, TA_1B_0, TA_0B_0\},$$

where the outcomes are sequences of three symbols with the following meanings:

- The first symbol is H or T, representing whether the coin lands heads or tails;
- The second symbol is either A_1 or A_0; A_1 implying that A makes the correct prediction, and A_0 implying that A makes the wrong prediction;
- The third symbol is either B_1 or B_0, with meanings analogous to those of A_1 and A_0, respectively.

For example, an outcome TA_1B_0 means that the coin lands tails, the scientist A makes the correct prediction, and the scientist B makes the wrong prediction.

The probability function $\mathbb{P}_1$ is defined as

$$\mathbb{P}_1(HA_1B_1) = 0.5, \quad \mathbb{P}_1\left(TA_1B_1\right) = 0.1,$$
$$\mathbb{P}_1\left(TA_1B_0\right) = 0.2, \quad \mathbb{P}_1\left(TA_0B_0\right) = 0.2.$$

Note that, with Ω_1 as the sample space, we have $E = \{TA_1B_0\}$ and $L_H \cap E = \emptyset$, which implies that

$$\mathbb{P}_1\left(L_H \middle| E\right) = 0.$$

We now build another pair $(\Omega_2, \mathbb{P}_2)$ of a sample space and a probability measure for which the conditional probability $\mathbb{P}_2\left(L_H \middle| E\right)$ is equal to 1. The sample space Ω_2 is defined as

$$\Omega_2 = \left\{HA_1B_0, HA_0B_1, HA_0B_0, TA_1B_1\right\}.$$

The probability function $\mathbb{P}_2$ is defined as

$$\mathbb{P}_2\left(HA_1B_0\right) = 0.3, \quad \mathbb{P}_2\left(HA_0B_1\right) = 0.1,$$
$$\mathbb{P}_2\left(HA_0B_0\right) = 0.1, \quad \mathbb{P}_2\left(TA_1B_1\right) = 0.5.$$

The events E and $L_H \cap E$, as subsets of Ω_2, contain a single outcome, HA_0B_1. Thus,

$$\mathbb{P}_2\left(L_H \middle| E\right) = 1. \quad \square$$

Question 6. A player chooses a number $k \leq 52$ and the top k cards are drawn one by one from a properly shuffled standard deck of 52 cards. The player wins if the last drawn card is an Ace and if there is exactly one more Ace among the cards drawn. Which k should the player choose to maximize the chance of winning in this game?

Answer: Denote by W_k the event that the k-th card is an Ace and that one more card is an Ace among the first $k-1$ cards. Note that $k \leq 50$ since there must be at least two cards (Aces) left after the top 50 cards are drawn. We need to find $1 \leq k \leq 50$ such that $\mathbb{P}(W_k)$ is maximized. Let N_k be the number of ways in which the deck can be shuffled to have one Ace in the k-th position, one Ace in a position from the set $\{1, 2, \ldots, k-1\}$, and two Aces in positions from the set $\{k+1, \ldots, 52\}$. Then,

$$\mathbb{P}(W_k) \;=\; \frac{N_k}{52!}. \tag{2.5}$$

Denote by M_k the number of ways in which we can choose the positions for four Aces. For each choice of positions for the four Aces, the Aces can be placed in those positions in 4! ways, while the other cards can be placed in the remaining positions in 48! ways. Therefore,

$$N_k \;=\; 4! \cdot 48! \cdot M_k. \tag{2.6}$$

To calculate M_k, note that: the k-th position must be chosen; one position is in $\{1,\ 2,\ \ldots,\ k-1\}$ and can be chosen in $k-1$ ways; the other two positions are in $\{k+1,\ k+2,\ \ldots,\ 52\}$ and can be chosen in $\binom{52-k}{2}$ ways. Thus,

$$\begin{aligned} M_k &= (k-1) \cdot \binom{52-k}{2} \\ &= \frac{(k-1)(52-k)(51-k)}{2}. \end{aligned} \tag{2.7}$$

From (2.5–2.7), it follows that the probability of winning, when choosing k, is

$$\begin{aligned} \mathbb{P}(W_k) &= \frac{4! \cdot 48!}{52!} \cdot \frac{(k-1)(52-k)(51-k)}{2} \\ &= \frac{12(k-1)(52-k)(51-k)}{52 \cdot 51 \cdot 50 \cdot 49.} \end{aligned}$$

The maximum value of $\mathbb{P}(W_k)$ is obtained for the value of k that maximizes the function

$$\begin{aligned} f(k) &= (k-1)(52-k)(51-k) \\ &= (k-1) \cdot [51-(k-1)] \cdot [50-(k-1)]. \end{aligned}$$

Let $z = k - 1$, $M = 51$, and

$$g(z) \; = \; z(51 - z)(50 - z).$$

Since $f(k) = g(k-1)$, the maximum of f is attained at $k_\star = z_\star + 1$, where $0 \leq z_\star \leq 49$ is the integer for which the function g is the largest; recall that $1 \leq k \leq 50$. The derivative of $g(z)$ is

$$g'(z) \; = \; 3z^2 - 2(2M-1)z + M(M-1)$$

and the solutions of the equation $g'(z) = 0$ are

$$z_{1,2} \quad = \quad \frac{2M - 1 \pm \sqrt{M^2 - M + 1}}{3}.$$

Since $(M-1)^2 < M^2 - M + 1 < M^2$, we find that

$$z_1 \in \left(\frac{M-1}{3}, \frac{M}{3} \right) = (16.67, 17)\,,$$
$$z_2 \in \left(M - \frac{2}{3}, M - \frac{1}{3} \right) = (50.33, 50.67)\,.$$

The quadratic function $g'(z)$ is negative on the interval (z_1, z_2) and positive everywhere else. This implies that g is increasing on $(-\infty, z_1)$, decreasing on (z_1, z_2), and increasing on $(z_2, +\infty)$. Hence, the value $0 \leq z_\star \leq 49$ where g attains its maximum is either $\lfloor z_1 \rfloor = 16$ or $\lceil z_1 \rceil = 17$. In other words, $z_\star \in \{16, 17\}$, which corresponds to $k_\star \in \{17, 18\}$. Since $f(17) = 19040$ and $f(18) = 19072$, we conclude that it is optimal to choose is $k = 18$. □

Question 7. Let N be a random variable whose values are positive integers. Prove that

$$\mathbb{E}\left[N\right] \; = \; \sum_{i=0}^{\infty} \mathbb{P}\left(N > i\right).$$

Answer: By definition,

$$\mathbb{E}[N] \;=\; \sum_{i=1}^{\infty} i \cdot \mathbb{P}[N = i]\,. \tag{2.8}$$

We replace the term $i \cdot \mathbb{P}[N = i]$ in (2.8) with i terms, each being equal to $\mathbb{P}[N = i]$. With this modification, the formula (2.8) becomes

$$\begin{array}{cccccccc} & \mathbb{E}[N] & & & & & & \\ = & \mathbb{P}[N=1] & & & & & & \\ + & \mathbb{P}[N=2] & + & \mathbb{P}[N=2] & & & & \\ + & \mathbb{P}[N=3] & + & \mathbb{P}[N=3] & + & \mathbb{P}[N=3] & & \\ & \vdots & & \vdots & & \vdots & & \ddots \end{array}$$

Note that the sum of the probabilities in the first column of the expression above is equal to $\mathbb{P}[N > 0]$. Similarly, the sum of the probabilities in the second column is equal to $\mathbb{P}[N > 1]$, and so on. Thus,

$$\mathbb{E}[N] = P[N > 0] + P[N > 1] + P[N > 2] + \ldots,$$

and therefore we conclude that

$$\mathbb{E}[N] \;=\; \sum_{i=0}^{\infty} \mathbb{P}(N > i)\,. \quad \square \tag{2.9}$$

Question 8. Each box of cereal contains a coupon. If there are p kinds of coupons, how many boxes of cereal have to be bought on average to obtain at least one coupon of each kind?

Answer: This is a well-known coupon collector problem. Denote by N the random variable representing the number of boxes of cereal that have to be bought to obtain all p different coupons. For $1 \leq k \leq p$, denote by N_k

the number of boxes that are bought from the time when $k-1$ different coupons were collected until the time when k different coupons have been collected. Note that $N_1 = 1$, since the first coupon is obtained by buying the first box. Then,

$$N = \sum_{k=1}^{p} N_k. \tag{2.10}$$

Since N_k has positive integer values, we can use (2.9) to obtain that

$$\mathbb{E}[N_k] = \sum_{i=0}^{\infty} \mathbb{P}(N_k > i). \tag{2.11}$$

The event $N_k > i$ occurs if and only if in each of the i boxes, that were bought after $k-1$ different coupons were collected, we only find one of the $k-1$ kinds of coupons already collected. Thus,

$$\mathbb{P}(N_k > i) = \left(\frac{k-1}{p}\right)^i, \quad \forall\, i \geq 0. \tag{2.12}$$

From (2.11) and (2.12), it follows that,

$$\begin{aligned} \mathbb{E}[N_k] &= \sum_{i=0}^{\infty} \left(\frac{k-1}{p}\right)^i = \frac{1}{1 - \frac{k-1}{p}} \\ &= \frac{p}{p+1-k}, \quad \forall\, 1 \leq k \leq p. \end{aligned} \tag{2.13}$$

From (2.10) and (2.13), we obtain that

$$\begin{aligned} \mathbb{E}[N] &= \sum_{k=1}^{p} \mathbb{E}[N_k] = \sum_{k=1}^{p} \frac{p}{p+1-k} \\ &= p \sum_{j=1}^{p} \frac{1}{j}, \end{aligned} \tag{2.14}$$

where, for the last equality, we used the substitution $j = p+1-k$.

For large values of p, an estimate for (2.14) can be provided by recalling that

$$\sum_{j=1}^{n} \frac{1}{j} \approx \ln n + \gamma + \frac{1}{2n},$$

as $n \to \infty$, where $\gamma \approx 0.5772$ is the Euler constant. Thus,

$$\mathbb{E}[N] \approx p \ln p + 0.5772p + \frac{1}{2}. \quad \Box \tag{2.15}$$

Remark. A popular interview question asks for the expected number of rolls of a standard die until each of the numbers appears at least once. Note that this problem is a special case of the coupon collector problem, as each roll of a die corresponds to opening a box (of cereal) containing a coupon from the set $\{1, 2, \ldots, 6\}$.

In particular, here $p = 6$, and (2.14) becomes

$$\mathbb{E}[N] = 6\left(1 + \frac{1}{2} + \cdots + \frac{1}{6}\right) = 14.70.$$

We conclude that the die has to be rolled on average 14.70 times for each number to appear at least once.

Also, note that (2.15) is a very robust approximation: for $p = 6$, the approximate value given by (2.15) is 14.71, while the exact value is 14.70.

Question 9. You roll a fair n-sided die repeatedly and sum the outcomes. What is the expected number of rolls until the sum is a multiple of n for the first time?

Answer: Denote by N the number of rolls until the sum is a multiple of n for the first time. Let X_i be the number rolled on the i-th roll and let $S_k = X_1 + \ldots + X_k$ be the sum of the numbers obtained in rolls 1 through k.

Fix $k \geq 1$. If S_k is not a multiple of n, then $S_k \equiv i \pmod{n}$ for some i with $1 \leq i \leq n-1$, and $S_{k+1} = S_k + X_{k+1}$ is a multiple on n only if $X_{k+1} = n - i$. In other words, there is only one number you can roll in the next roll that would make S_{k+1} a multiple of n. Hence,

$$\mathbb{P}\left(n \nmid S_{k+1} \mid n \nmid S_k\right) = 1 - \frac{1}{n}.$$

We also have

$$\mathbb{P}\left(n \nmid S_1\right) = \mathbb{P}\left(n \nmid X_1\right) = 1 - \frac{1}{n}.$$

Therefore,

$$\begin{aligned} \mathbb{P}(N > i) &= \mathbb{P}\left(n \nmid S_1\right) \cdot \prod_{k=1}^{i-1} \mathbb{P}\left(n \nmid S_{k+1} \mid n \nmid S_k\right) \\ &= \left(1 - \frac{1}{n}\right)^i. \qquad (2.16) \end{aligned}$$

Since N has positive integer values, we can use (2.9) and (2.16) to obtain that

$$\begin{aligned} \mathbb{E}[N] &= \sum_{i=0}^{\infty} \mathbb{P}(N > i) \\ &= \sum_{i=0}^{\infty} \left(1 - \frac{1}{n}\right)^i \\ &= \frac{1}{1 - \left(1 - \frac{1}{n}\right)} \\ &= n. \quad \square \end{aligned}$$

Question 10. Is it possible to have two non-fair 6-sided dice, with sides numbered 1 through 6, with a uniform sum probability?

Answer: It is not possible to construct two dice with those properties. We will show this by contradiction: assume that probabilities $p_{i,j}$ exist, with $i \in \{1,2\}$, $j \in \{1,2,\ldots,6\}$, where $p_{i,j}$ denotes the probability of rolling j with the i-th dice. Denote by Σ the sum of the numbers obtained in a single roll of these two dice. Then, the assumption of the uniform sum probability implies that

$$\mathbb{P}(\Sigma = 2) = p_{1,1}p_{2,1} = \frac{1}{11}; \quad (2.17)$$

$$\mathbb{P}(\Sigma = 12) = p_{1,6}p_{2,6} = \frac{1}{11}. \quad (2.18)$$

By using the arithmetic – geometric means inequality, (2.17), and (2.18), we obtain that

$$\begin{aligned} \mathbb{P}(\Sigma = 7) &\geq p_{1,6}p_{2,1} + p_{1,1}p_{2,6} \\ &\geq 2\sqrt{p_{1,6}p_{2,1}p_{1,1}p_{2,6}} \\ &= \frac{2}{11}. \end{aligned}$$

Thus, $P(\Sigma = 7) \neq \frac{1}{11}$, which contradicts the assumption of the uniform sum probability. □

Question 11. Consider 2^n players of equal skill[2] playing a game where the players are paired off against each other at random. The 2^{n-1} winners are again paired off randomly, and so on until a single winner remains. Find the probability that two contestants never play each other.

Answer: Denote the contestants by $C_1, C_2, \ldots, C_{2^n}$. Let $T_{i,j}$ be the event that C_i and C_j play each other in the tournament, where $1 \leq i < j \leq 2^n$. Due to symmetry, all of these events have equal probabilities. Hence, we can assume that our two contestants in question are $A = C_1$

[2] The probability of winning a game between any two players is $\frac{1}{2}$ for each player.