Ten Sample Questions

Challenging Brainteasers for Interviews

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"Aha" Questions

Question 9. Bob is a witty trader who trades exotic fruit grown far away. He travels from one place to another with three sacks which can hold 30 fruits each. None of the sacks can hold more than 30 fruits. He starts with 30 fruits in each sack. On his way, he must pass through 30 checkpoints, and at each checkpoint he has to give one fruit for each sack to the authorities. How many fruits remain after he goes through all the 30 checkpoints?

Logical Conundrums

Question 13. An evil troll once captured a bunch of gnomes and told them: "Tomorrow, I will make you stand in a file, one behind the other, ordered by height such that the tallest gnome can see everybody in front of him. I will place either a white cap or a black cap on each head. Then, starting from the tallest, each gnome has to declare aloud what he thinks the color of his own cap is. In the end, those who were correct will be spared; the others will be eaten, silently."

The gnomes thought about it and came up with an optimal strategy. How many of them survived? What if the hats come in 10 different colors?

Discrete Probability

Question 15. How many times do I have to roll a die on average until I roll the same number six times in a row?

Continuous Probability

Question 12. An infinite sheet of paper has inscribed on it a set of horizontal lines D units apart and a set of vertical lines D units apart. A needle of length L (where $L < D$) is twirled and tossed on the paper. What is the expected number of lines crossed by the needle? What is the probability that the needle crosses a line?

Counting Challenges

Question 10. In how many ways can you divide 7 candies and 14 stickers among 4 children such that each child gets at least one candy and also gets more stickers than candies?

Games and Invariants

Question 16. A room has n computers, less than half of which are damaged. It is possible to query a computer about the status of any computer. A damaged computer could give wrong answers. How can you discover an undamaged computer?

Number Theory

Question 2. You are given three piles with 5, 49, and 51 pebbles respectively. Two operations are allowed:
(a) merge two piles together, or
(b) divide a pile with an even number of pebbles into two equal piles.
Is there a sequence of operations that would result in 105 piles with one pebble each?

Geometry

Question 4. The new campus of University College is a perfect disk of radius 1km. The Coffee Company plans to open 7 coffee shops on campus. Where do they have to be placed in order to minimize the maximum (straight-line) distance that a person anywhere on the campus has to walk to find a coffee shop?

Calculus

Question 9. Compute the integral

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx.$$

Linear Algebra

Question 4. A bug starts at the vertex A of a triangle ABC. It then moves to one of its two adjacent vertices. How many paths of length 8 end back at vertex A? For example, one such path is $ABCABCABA$.

Solutions – Ten Sample Questions

"Aha" Questions

Question 9. Bob is a witty trader who trades exotic fruit grown far away. He travels from one place to another with three sacks which can hold 30 fruits each. None of the sacks can hold more than 30 fruits. He starts with 30 fruits in each sack. On his way, he must pass through 30 checkpoints, and at each checkpoint he has to give one fruit for each sack to the authorities. How many fruits remain after he goes through all the 30 checkpoints?

Answer: While Bob has fruits in all three sacks, he will have to give three fruits at each checkpoint. The optimal approach for Bob is to try to empty one bag as quickly as possible, so he will only have to give two fruits from that point on, then focus on emptying a second bag, after which he would only have to give one fruit at each checkpoint.
In other words, when passing through the first 10 checkpoints, Bob will give three fruits at each checkpoint from the same bag until that bag is empty. He is now left with two bags with 30 fruits each. When passing through the next 15 checkpoints, Bob will give two fruits at each checkpoint from the same bag until that bag is empty. He now has 30 fruits left, all of them in one bag. Bob has to go through 5 more checkpoints and will give one fruit at each of those checkpoints.
Bob will be left with 25 fruits, all of them in one bag. □

Logical Conundrums

Question 13. An evil troll once captured a bunch of gnomes and told them: "Tomorrow, I will make you stand in a file, one behind the other, ordered by height such that the tallest gnome can see everybody in front of him. I will place either a white cap or a black cap on each head. Then, starting from the tallest, each gnome has to declare aloud what he thinks the color of his own cap is. In the end, those who were correct will be spared; the others will be eaten, silently."

The gnomes thought about it and came up with an optimal strategy. How many of them survived? What if the hats come in 10 different colors?

Answer: We will show that there is a strategy that saves everyone except for the tallest gnome. Observe that the problem is equivalent to the one in which gnomes hats have numbers instead of colors. If there are k different colors, then the gnomes should replace them with the numbers 0, 1, $\dots$, $k-1$. If there are only two colors, the gnomes should replace them with the numbers 0 and 1. The tallest gnome should sum up all the numbers, calculate its remainder when divided by k, and announce the result to the evil troll. The other gnomes will hear the remainder of the total sum. The second to the tallest gnome will be able to figure out the number that correspond to his hat. After saying that number, the next gnome (third tallest) is in an equivalent position to the second tallest gnome. He can see all the hats in front of him, he has heard the total sum, and he has heard the number on the hat from the gnome behind him. This third tallest gnome can correctly say the number on his hat. The next in line does the same. The gnomes continue with this strategy and all of them are saved, except for the tallest one. $\square$

Discrete Probability

Question 15. How many times do I have to roll a die on average until I roll the same number six times in a row?

Answer: Let E_n be the expected number of rolls until you roll the same number n times in a row. We need to find E_6. Assume that you have just rolled the same number $n-1$ times in a row, which took E_{n-1} rolls on average. Then, one of the following two outcomes will happen:

• with probability $\frac{1}{6}$, you roll the same number again on the next roll; in this case, you end up rolling the same number n times in a row and it took $E_{n-1} + 1$ rolls to obtain this outcome;
• with probability $\frac{5}{6}$, you roll a different number on the next roll; in this case, you already started on your way to roll the same number n times in a row and you will need $E_{n-1} + E_n$ to achieve this.
Hence, we obtain the following recurrence:

$$E_n = \frac{1}{6}(E_{n-1}+1) + \frac{5}{6}(E_{n-1}+E_n),$$

which is equivalent to

$$E_n = 6E_{n-1}+1, \quad \forall\, n > 1. \tag{1}$$

By iterating (3.49), we obtain that

$$E_n = 1 + 6 + \ldots + 6^{n-2} + 6^{n-1}E_1. \tag{2}$$

Since $E_1 = 1$, we obtain from (3.50) that

$$E_n = \sum_{i=0}^{n-1} 6^i = \frac{6^n - 1}{5}.$$

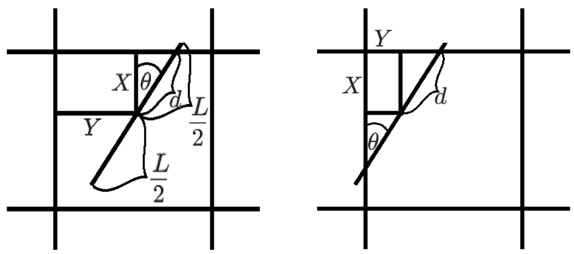
Thus, for $n = 6$, we find that $E_6 = 9331$. □

Continuous Probability

Question 12. An infinite sheet of paper has inscribed on it a set of horizontal lines D units apart and a set of vertical lines D units apart. A needle of length L (where $L < D$) is twirled and tossed on the paper. What is the expected number of lines crossed by the needle? What is the probability that the needle crosses a line?

Answer: Since $L < D$, the needle can intersect at most one horizontal and at most one vertical line. Denote by

X the distance from the midpoint of the needle to the closest horizontal line.[1] Denote by Y the distance from the midpoint of the needle to the closest vertical line. Let θ denote the angle between the vertical line through the midpoint of the needle and the line containing the needle itself, as in the figure below.



Note that the position of the needle is completely determined by the values of X, Y, and θ. The random variables X, Y, and θ are independent; X and Y are uniformly distributed on $\left(0, \frac{D}{2}\right)$, while θ is uniformly distributed on $\left(0, \frac{\pi}{2}\right)$. Denote by A the event that the needle crosses a horizontal line and by B the event that the needle crosses a vertical line.
Since the needle can intersect at most one horizontal and at most one vertical line, the expected number of lines crossed by the needle is precisely

$$\mathbb{P}(A) + \mathbb{P}(B) = 2\,\mathbb{P}(A).$$

The last equality follows from $\mathbb{P}(A) = \mathbb{P}(B)$, a fact easily established by rotating the sheet of paper by $\pi/2$. Simi-

[1] The probability that the needle will land on the sheet of paper so that its midpoint is exactly halfway between two consecutive horizontal lines is zero, so the horizontal line closest to the midpoint of the needle is well-defined.

larly, the probability that the needle crosses a line is precisely

$$\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B) \;=\; 2\,\mathbb{P}(A)-\mathbb{P}(A\cap B)\,.$$

Hence, we need to compute $\mathbb{P}(A)$ and $\mathbb{P}(A\cap B)$. Denote by d the distance from the midpoint of the needle to the point of intersection of the line containing the needle itself with the horizontal line (see figure).

Note that $d=\frac{X}{\cos\theta}$. The event A occurs if and only if $d<\frac{L}{2}$, that is, if and only if $\frac{X}{\cos\theta}<\frac{L}{2}$. Similarly, the event B occurs if and only if $\frac{Y}{\sin\theta}<\frac{L}{2}$.

Since X and θ are independent, their joint density $f_{X,\theta}$ is the product of marginal densities and, thus, satisfies

$$f_{X,\theta}(u,v)=\frac{1}{D/2}\cdot\frac{1}{\pi/2}.$$

By integrating the joint density over an appropriate region, we obtain that

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}\left(\frac{X}{\cos\theta}<\frac{L}{2}\right)=\mathbb{P}\left(X<\frac{L}{2}\cos\theta\right)\\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{L}{2}\cos\theta}\frac{1}{D/2}\cdot\frac{1}{\pi/2}\,dx\,d\theta\\
&= \frac{4}{\pi D}\int_0^{\frac{\pi}{2}}\frac{L}{2}\cos\theta\,d\theta\\
&= \frac{2L}{\pi D}\sin\theta\Big|_0^{\pi/2}\\
&= \frac{2L}{\pi D}.
\end{aligned}$$

Similarly, since X, Y, and θ are independent, their joint density is the product of marginal densities and, thus, is equal to $\left(\frac{1}{D/2}\right)^2 \cdot \frac{1}{\pi/2}$. Then,

$$\begin{aligned}
&\mathbb{P}\left(A \cap B\right) \\
= \;& \mathbb{P}\left(\frac{X}{\cos\theta} < \frac{L}{2} \text{ and } \frac{Y}{\sin\theta} < \frac{L}{2}\right) \\
= \;& \mathbb{P}\left(X < \frac{L}{2}\cos\theta \text{ and } Y < \frac{L}{2}\sin\theta\right) \\
= \;& \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\cos\theta} \int_0^{\frac{L}{2}\sin\theta} \left(\frac{1}{D/2}\right)^2 \cdot \frac{1}{\pi/2}\, dy\, dx\, d\theta.
\end{aligned}$$

The last integral can be evaluated as follows:

$$\begin{aligned}
\mathbb{P}\left(A \cap B\right) &= \frac{8}{\pi D^2} \cdot \frac{L^2}{4} \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \\
&= \frac{L^2}{\pi D^2} \int_0^{\frac{\pi}{2}} \sin\left(2\theta\right) d\theta \\
&= \frac{L^2}{\pi D^2} \cdot \left(-\frac{1}{2}\cos\left(2\theta\right)\right)\Big|_0^{\pi/2} \\
&= \frac{L^2}{\pi D^2}.
\end{aligned}$$

We conclude that the expected number of lines crossed by the needle is

$$2\,\mathbb{P}\left(A\right) = \frac{4L}{\pi D},$$

while the probability that the needle crosses a line is

$$2\,\mathbb{P}\left(A\right) - \mathbb{P}\left(A \cap B\right) = \frac{4L}{\pi D} - \frac{L^2}{\pi D^2}. \quad \square$$

Counting Challenges

Question 10. In how many ways can you divide 7 candies and 14 stickers among 4 children such that each child gets at least one candy and also gets more stickers than candies?

Answer: First, we distribute the candies. Let c_i denote the number of candies the ith child gets, $i \in \{1, 2, 3, 4\}$. Counting the ways to distribute 7 candies among 4 children, so that each child gets at least one candy, is then equivalent to counting positive integer solutions to equation

$$c_1 + c_2 + c_3 + c_4 = 7.$$

Any such solution, say $(c_1, c_2, c_3, c_4) = (1, 2, 3, 1)$ can be uniquely represented as a sequence

$$\star \Big| \star\star \Big| \star\star\star \Big| \star$$

of 7 stars (representing candies) and 3 bars (separating the candy lots that 4 children receive). So, the number of positive integer solutions to equation $c_1 + c_2 + c_3 + c_4 = 7$ is precisely the number of sequences of 7 stars and 3 bars, where bars are placed in between the stars, but not before the first or after the last star, and so that no place between two stars can be occupied by more than one bar (since each child gets at least one candy). Hence, one only needs to choose 3 out of the 6 places between the stars and place bars there. Thus, the number of ways to distribute 7 candies among 4 children, so that each child gets at least one candy, is $\binom{6}{3}$.

Once we have distributed the candies, we need to distribute the stickers so that each child gets more stickers than candies. Let s_i denote the number of stickers the ith child gets, $i = 1, 2, 3, 4$. Counting integer solutions to equation $s_1 + s_2 + s_3 + s_4 = 14$ such that $s_i > c_i$ is equivalent to counting positive integer solutions to equation $(s_1 - c_1) + (s_2 - c_2) + (s_3 - c_3) + (s_4 - c_4) = 14 - 7 = 7$. Thus, setting $d_i = s_i - c_i$, we need to count positive integer solutions to equation $d_1 + d_2 + d_3 + d_4 = 7$, which is precisely the counting problem we have already solved in the previous paragraph. Thus, (with candies already distributed among the children) the number of ways to

distribute the stickers so that each child gets more stickers than candies is $\binom{6}{3}$.
We conclude that the number of ways in which we can divide 7 candies and 14 stickers among 4 children, so that each child gets at least one candy and also gets more stickers than candies, is

$$\binom{6}{3}^2 = 400. \quad \Box$$

Games and Invariants

Question 16. A room has n computers, less than half of which are damaged. It is possible to query a computer about the status of any computer. A damaged computer could give wrong answers. How can you discover an undamaged computer?

Answer: A computer replies with "damaged" or "undamaged" to a query on the status of another computer. Our algorithm for discovering an undamaged computer in the room is based on the following observation: Assuming that there is a linked list of computers where each computer except the last is queried on the status of its successor in the list, if the replies to all queries are "undamaged" and if there is at least one undamaged computer in the list, then the last computer in the list is definitely undamaged! Our algorithm constructs such a linked list as detailed below.
Start with an empty list. Repeatedly affix one of the remaining computers to the end of the list. If the resulting list has at least two computers and if the penultimate computer replies with "damaged" to the query on the status of the last computer in the list, then remove both of these computers from the list and discard them from the room forever. At least one of the two computers discarded is damaged, so undamaged computers in the room remain in

the majority throughout the process of building the linked list. The linked list must be non-empty after all computers have been processed, and there must be at least one undamaged computer in that list. Then, the last computer in the list must be undamaged. The total number of queries is exactly $n-1$. □

Number Theory

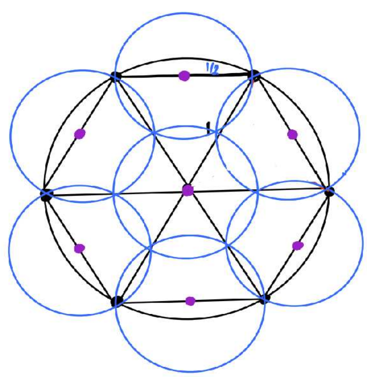
Question 2. You are given three piles with 5, 49, and 51 pebbles respectively. Two operations are allowed:
(a) merge two piles together, or
(b) divide a pile with an even number of pebbles into two equal piles.
Is there a sequence of operations that would result in 105 piles with one pebble each?

Answer: No! Each of the three piles at the beginning has an odd size. Hence, the first operation must be (a), that is, we must merge two existing piles together, resulting in two piles with possible sizes $\{51, 54\}$, $\{5, 100\}$, or $\{49, 56\}$. Hence, after the first operation, we end up with an even-sized pile and an odd-sized pile.
Denote by d the greatest common divisor of these two pile sizes. In particular, $d = 3$ if the piles are $\{51, 54\}$; $d = 5$ if the piles are $\{5, 100\}$; and $d = 7$ if the piles are $\{49, 56\}$.
Note that, since d is odd and greater than 1, no matter what sequence of operations is carried out (merging two existing piles or dividing an even–sized pile into two equal piles), the size of every pile size will continue to be a multiple of d. Since $d > 1$, we can never reach a configuration where all piles are of size 1. □

Geometry

Question 4. The new campus of University College is a perfect disk of radius 1km. The Coffee Company plans to open 7 coffee shops on campus. Where do they have to be placed in order to minimize the maximum (straight-line) distance that a person anywhere on the campus has to walk to find a coffee shop?

Answer: In order to minimize the maximum distance to a coffee shop, the Coffee Company should place the seven coffee shops at the midpoints of the sides and at the center of the regular hexagon inscribed in the perfect disk of radius 1km, as indicated in the figure below. This way, the maximum distance a person anywhere on campus has to walk to find a coffee shop is exactly 0.5km.



Why is 0.5km optimal?
Suppose that, on the contrary, the Coffee Company could place seven coffee shops on campus so that the the maximum distance from anywhere on campus to a coffee shop would be $r < 0.5$km. Then, the seven disks of radius r, centered at those coffee shops, would cover the entire campus, that is, the entire disk of radius of 1km.
Let us prove that a disk of radius r ($r < 1$) can cover at most $2\sin^{-1}(r)$ of the circumference of the disk of radius 1. Let us denote by A and B the instersections of the two

disks. Then $AB \leq 2r$. If O Is the center of the big disk of radius 1 and M the midpoint of AB, then we have

$$AM = \sin \angle AOM.$$

The arc AB has the length equal to $\angle AOB$. This length satisfies

$$\angle AOB = 2\angle AOM = 2\sin^{-1}(AM) \leq 2\sin^{-1}(r).$$

Then, six disks of radius r cover at most $6 \cdot 2\sin^{-1}(r) = 12\sin^{-1}(r)$ of the circumference of the disk of radius 1. Note that, since $r < 0.5$, it follows that $\sin^{-1}(r) < \frac{\pi}{6}$ and therefore

$$12 \cdot \frac{\pi}{6} \; = \; 2\pi.$$

Thus, six of the disks of radius r cannot cover the entire circumference of the unit circle, This means that all of the seven disks of radius r must touch the circumference of the disk of radius 1. Since $r < 0.5$, none of those disks would cover the center the disk of radius 1; in other words, a person at the center of the campus would have to walk more than 0.5km to the closest coffee shop, which is a contradiction. $\square$

Calculus

Question 9. Compute the integral

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx.$$

Answer: Let

$$\mathcal{I} \; = \; \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx. \tag{3}$$

We will use the substitution $u = \frac{\pi}{2} - x$. Then, $x = 0$ corresponds to $u = \frac{\pi}{2}$, $x = \frac{\pi}{2}$ corresponds to $u = 0$,

$x = \frac{\pi}{2} - u$, and therefore $dx = -du$. Thus,

$$\begin{aligned}
\mathcal{I} &= \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(x\right)\right) dx \\
&= \int_{\frac{\pi}{2}}^{0} \ln\left(\cos\left(u\right)\right) \left(-du\right) \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(u\right)\right) du \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(x\right)\right) dx, \qquad (4)
\end{aligned}$$

where for the last equality we changed the integration variable from u to x.

By adding (5.40) and (5.41), we obtain that

$$\begin{aligned}
2\mathcal{I} &= \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(x\right)\right) + \ln\left(\cos\left(x\right)\right) dx \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\sin\left(x\right)\cos\left(x\right)\right) dx. \qquad (5)
\end{aligned}$$

Recall that $\sin(2x) = 2\sin(x)\cos(x)$ and write (5.42) as

$$\begin{aligned}
2\mathcal{I} &= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin\left(2x\right)\right) dx \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\right) dx \;+\; \int_0^{\frac{\pi}{2}} \ln\left(\sin(2x)\right) dx \\
&= -\frac{\pi}{2}\ln 2 \;+\; \int_0^{\frac{\pi}{2}} \ln\left(\sin(2x)\right) dx, \qquad (6)
\end{aligned}$$

since $\ln\left(\frac{1}{2}\right) = -\ln 2$.

By using the substitution $w = 2x$, we find that

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \ln(\sin(2x))\, dx \\
&= \frac{1}{2}\int_0^{\pi} \ln(\sin(w))\, dw \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin(w))\, dw + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(w))\, dw \\
&= \frac{\mathcal{I}}{2} + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(w))\, dw \qquad (7)
\end{aligned}$$

Furthermore, by using the substitution $u = \pi - w$, we obtain that

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(w))\, dw \\
&= \int_{\frac{\pi}{2}}^{0} \ln(\sin(\pi - u))\,(-du) \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin(u))\, du \\
&= \mathcal{I}, \qquad (8)
\end{aligned}$$

where we used the fact that $\sin(\pi - u) = \sin(u)$.
From (5.44) and (5.45), it follows that

$$\int_0^{\frac{\pi}{2}} \ln(\sin(2x))\, dx = \frac{\mathcal{I}}{2} + \frac{\mathcal{I}}{2} = \mathcal{I}, \qquad (9)$$

and, from (5.43) and (5.46), we conclude that

$$2\mathcal{I} = -\frac{\pi}{2}\ln 2 + \mathcal{I}.$$

Thus, $\mathcal{I} = -\frac{\pi}{2}\ln 2$ and therefore

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = -\frac{\pi}{2}\ln 2. \quad \square$$

Linear Algebra

Question 4. A bug starts at the vertex A of a triangle ABC. It then moves to one of its two adjacent vertices. How many paths of length 8 end back at vertex A? For example, one such path is $ABCABCABA$.

Answer: The triangle on which the bug moves is a complete graph of size 3. Denote by $E = (e_{ij})_{1\leq i,j\leq 3}$ the 3×3 incidence matrix of this graph. The entries of the matrix E are 0 or 1. The entry e_{ij} is 1 if there is an edge between the vertices i and j. Re–label the vertices A, B, and C of the triangle to 1, 2, and 3. The incidence matrix is

$$E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We will use the following result which was also used in Question 2:

If E is an incidence matrix of a graph, then $E^k(i,j)$ is the total number of paths of length k from the vertex i to the vertex j.

Thus, the number of paths of length 8 starting at A and ending back at A is equal to the entry $(1,1)$ of the matrix E^8.

To calculate E^8, note that, since the matrix E is symmetric, it has the diagonal form

$$E = QDQ^t \tag{10}$$

where Q is the orthogonal matrix whose columns are the eigenvectors of the matrix E, and D is a diagonal matrix whose diagonal entries are the eigenvalues of E. Then,

$$E^8 = QD^8Q^t. \tag{11}$$

We now compute the matrices D and Q. The characteristic polynomial of the matrix E is

$$\begin{aligned} P_E(t) &= \det(tI - E) = \det \begin{bmatrix} t & -1 & -1 \\ -1 & t & -1 \\ -1 & -1 & t \end{bmatrix} \\ &= t^3 - 2 - 3t = (t-2)(t+1)^2. \end{aligned}$$

The roots of $P_E(t)$ are 2, with multiplicity 1 and -1, with multiplicity 2. Since the eigenvalues of a matrix are the roots of its characteristic polynomial, it follows that the eigenvalues of the matrix E are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = -1$. The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}; \quad v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}; \quad v_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}.$$

Thus, the matrices Q and D are

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}; \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then, from (5.126), we obtain that

$$E^8 = Q \begin{bmatrix} 2^8 & 0 & 0 \\ 0 & (-1)^8 & 0 \\ 0 & 0 & (-1)^8 \end{bmatrix} Q^t.$$

By direct computation, we find that $E^8(1,1) = 86$ and therefore conclude that there are 86 paths of length 8 that start at the vertex A and end back at A. $\square$